CHAPTER 14

Chapter 14

Exercise 14.1

This example will be skipped as said by Dr. Emerson.

Exercise 14.2

Confirm that when Nielsen and Chuang define $\sigma_- = |0\rangle\langle 1|$ in the context of amplitude damping (viz, page 388 in the first edition), this is inconsistent with the usual definition $\sigma_\pm = \frac{\sigma_x \pm i\sigma_y}{2}$

Solution: Based on the definition of Nielsen & Chuang

$$\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$$

However, the usual definition gives

$$\sigma_- = \frac{\sigma_x - i\sigma_y}{2} = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = |1\rangle\langle 0|$$

We can see that the two definitions are inconsistent

Exercise 14.3

Calculate the Lindblad operators associated with phase-damping and show that resulting the master equation for phase damping is equivalent to (14.37).

Solution: The Kraus operators can be expanded as:

$$A_0 = \mathbb{1} - \frac{\lambda}{2}\sigma_-\sigma_+ + O(\Delta t^2) = A_0^\dagger, \quad A_1 = \sqrt{\lambda}\sigma_-\sigma_+ = A_1^\dagger$$

Therefore

$$\lim_{\Delta t \to 0} \frac{\rho(t+\Delta t) - \rho}{\Delta t} = \lim_{\Delta t \to 0} \frac{A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger - \rho}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\rho - \frac{\lambda}{2}(\rho\sigma_-\sigma_+ + \sigma_-\sigma_+\rho) + \lambda\sigma_-\sigma_+\rho\sigma_-\sigma_+ - \rho}{\Delta t} = -\frac{\Gamma_1}{2}\{\rho, \sigma_-\sigma_+\} + \Gamma_1\sigma_-\sigma_+\rho\sigma_-\sigma_+$$

Hence

$$\frac{d\rho}{dt} = -\frac{\Gamma_1}{2}\begin{pmatrix} 0 & \rho_{01} \\ \rho_{10} & 2\rho_{11} \end{pmatrix} + \Gamma_1 \begin{pmatrix} 0 & 0 \\ 0 & \rho_{11} \end{pmatrix} = -\frac{\Gamma_1}{2}\begin{pmatrix} 0 & \rho_{01} \\ \rho_1 & 0 \end{pmatrix}$$

which is equivalent to Equation 14.37

Exercise 14.4

Check that these three operators form a valid set of Kraus operators.

Solution: The operators must satisfy $\sum_k A_k^\dagger A_k = \mathbb{1}$. Note that

$$A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1-\lambda-\gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} = \mathbb{1}$$

Therefore, the three operators form a valid set of Kraus operators.

Exercise 14.5

Explain why we can compress the four Kraus operators associated with amplitude and phase damping, understood as independent processes, into only three Kraus operators in the context of the above argument.

Solution: The Kraus Representation of the composition of the Phase and Amplitude Damping channels can be achieved by multiplying the Kraus operators.

$$A'_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda(1-\gamma)-\gamma} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda'-\gamma} \end{pmatrix}$$

$$A'_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda(1-\gamma)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda'} \end{pmatrix}$$

$$A'_2 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$A'_3 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can see that the fourth Kraus operator is a zero matrix. Therefore, we only require three Kraus operators.

Exercise 14.6

Calculate the Lindblad operators associated with simultaneous amplitude and phase-damping and show that the resulting master equation is equivalent to that obtained from (14.39). In particular confirm whether the off-diagonal term should have the form $\rho_{01}e^{-1(\Gamma_1+\Gamma_2)t/2}$ or $\rho_{01}^{-(\Gamma_1+2\Gamma_2)t/2}$ as in (14.37).

Solution:

$$A_0 = \mathbb{1} - \frac{\gamma+\lambda}{2}\sigma_-\sigma_+ + O(\Delta t^2) = A_0^\dagger, \quad A_1 = \sqrt{\lambda}\sigma_-\sigma_+ = A_1^\dagger, \quad A_2 = \sqrt{\gamma}\sigma_+, A_2^\dagger = \sqrt{\gamma}\sigma_-$$

$$\lim_{\Delta t\to 0} \frac{\rho(t+\Delta t)-\rho}{\Delta t} = -\frac{\gamma+\lambda}{2}\{\rho, \sigma_-\sigma_+\} + \lambda\sigma_-\sigma_+\rho\sigma_-\sigma_+ + \gamma\sigma_+\rho\sigma_-$$

Hence

$$\frac{d\rho}{dt} = -\frac{\Gamma_1+\Gamma_2}{2}\begin{pmatrix} 0 & \rho_{01} \\ \rho_{01} & 2\rho_{11} \end{pmatrix} + \Gamma_1 \begin{pmatrix} 0 & 0 \\ 0 & \rho_{11} \end{pmatrix} + \Gamma \begin{pmatrix} \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma_1\rho_{11} & -\frac{\Gamma_1+\Gamma_2}{2}\rho_{01} \\ -\frac{\Gamma_1+\Gamma_2}{2}\rho_{01} & -\Gamma_1\rho_{11} \end{pmatrix}$$

Therefore, the off-diagonal term should have the form $\rho_{01}e^{-1(\Gamma_1+\Gamma_2)t/2}$.

Exercise 14.7

Verify the correctness of eqns. (14.43) and (14.44) – there are almost certainly some sign errors in the above – then recast eqns. (14.43) in terms of the timedependent expectation values for the spin components $(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$

Solution: The Hamiltonian is given by:

$$\begin{aligned}
H &= H_0 + H_1(t) \\
&= \frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\omega_1}{2}(\sigma_z \cos\omega t + \sigma_y \sin\omega t) \\
&= \frac{\hbar}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix} + \frac{\hbar\omega_1}{2}\begin{pmatrix} 0 & \cos\omega t - i\sin\omega t \\ \cos\omega t + i\sin\omega t & 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}
\end{aligned}$$

Assuming ρ is given by:

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

Using the Von Neumann equation, we gave

$$\begin{aligned}
i\hbar\frac{\partial\rho}{\partial t} &= [H, \rho] = H\rho - \rho H \\
&= \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \omega_0\rho_{00} + \omega_1 e^{-i\omega t}\rho_{10} & \omega_0\rho_{01} + \omega_1 e^{-i\omega t}\rho_{11} \\ \omega_1 e^{i\omega t}\rho_{00} - \omega_0\rho_{10} & \omega_1 e^{i\omega t}\rho_{01} - \omega_0\rho_{11} \end{pmatrix} \\
&- \frac{\hbar}{2}\begin{pmatrix} \omega_0\rho_{00} + \omega_1 e^{-i\omega t}\rho_{01} & \rho_{00}\omega_1 e^{-i\omega t} - \omega_0\rho_{01} \\ \omega_0\rho_{10} + \omega_1 e^{i\omega t}\rho_{11} & \omega_1 e^{i\omega t}\rho_{10} - \omega_0\rho_{11} \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \omega_1(e^{-i\omega t}\rho_{10} - \rho_{01}e^{-i\omega t}) & 2\omega_0\rho_{01} + \omega_1 e^{-i\omega t}(\rho_{00} - \rho_{11}) \\ \omega_1 e^{i\omega t}(\rho_{11} - \rho_{00}) - 2\omega_0\rho_{10} & \omega_1(e^{i\omega t}\rho_{01} - \rho_{10}e^{i\omega t}) \end{pmatrix}
\end{aligned}$$

Define $\Delta := \rho_{11}(t) - \rho_{00}(t)$, and $r = \rho_{01}$. Therefore

$$\begin{aligned}
\frac{\partial\Delta}{\partial t} = \frac{\partial\rho_{11}}{\partial t} - \frac{\partial\rho_{00}}{\partial t} &= -\frac{i}{2}\omega_1\left(e^{-i\omega t}\rho_{10} - e^{i\omega t}\rho_{01}\right) - \left(-i\omega_1 e^{i\omega t}\rho_{01} - e^{-i\omega t}\rho_{10}\right) \\
&= -i\frac{\omega_1}{2}\left(e^{-i\omega t}\rho_{10} - e^{i\omega t}\rho_{01}\right) - \left(\omega_1 e^{i\omega t}\rho_{01} - \omega_1 e^{-i\omega t}\rho_{10}\right) \\
&= \frac{-i}{2}\omega_1\left(2\omega_1 e^{-i\omega t}\rho_{10} - 2\omega_1 e^{i\omega t}\rho_{01}\right) \\
&= i\omega_1\left(re^{i\omega t} - r^* e^{-i\omega t}\right)
\end{aligned}$$

which has a different sign compared to Equation 14.43. Similarly

$$\begin{aligned}
\frac{\partial r}{\partial t} = \frac{\partial\rho_{01}}{\partial t}(t) &= \frac{-i}{2}\left(2\omega_0\rho_{01} + \omega_1 e^{-i\omega t}\left(\rho_{11} - \rho_{00}\right)\right) \\
&= -\frac{i}{2}\left(2\omega_0 r + \omega_1 e^{-i\omega t}\Delta\right) \\
&= -i\omega_0 r - \frac{i\omega_1}{2}\Delta e^{-i\omega t}
\end{aligned}$$

Adding the relaxation terms associated with the amplitude and phase damping model above, we get the Bloch equations

$$\begin{aligned}
\frac{\partial\Delta}{\partial t} &= -i\omega_1\left(r^* e^{-i\omega t} - re^{i\omega t}\right) - \Gamma_1\left(\Delta - \Delta^{eq}\right) \\
\frac{\partial r}{\partial t} &= -i\omega_0 r - \frac{i\omega_1}{2}\Delta e^{-i\omega t} - \Gamma_2 r
\end{aligned}$$

Defining $r = r' e^{-i\omega t} \Rightarrow r' = re^{i\omega t}$ gives

$$\frac{\partial\Delta}{\partial t} = -i\omega_1\left(r'^* - r'\right) - \Gamma_1\left(\Delta - \Delta^{eq}\right)$$

which also has an error sign compared to Equation 14.44.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(r' e^{-i\omega t}\right) &= -i\omega_0 r' e^{-i\omega t} - \frac{i\omega_1}{2}\Delta e^{-i\omega t} - \Gamma_2 r' e^{-i\omega t} \\
&= e^{-i\omega t}\frac{\partial r'}{\partial t} + r' e^{-i\omega t}\cdot(-i\omega) = \left(-i\omega_0 r' - \frac{i\omega_1}{2}\Delta - \Gamma_2 r'\right)e^{-i\omega t} \\
&= \frac{\partial r'}{\partial t} = -i\omega_0 r' - \frac{i\omega_1}{2}\Delta - \Gamma_2 r' + r' i\omega \\
&= i\left(\omega - \omega_0\right)r' - \frac{i\omega_1}{2}\Delta - \Gamma_2 r'
\end{aligned}$$

We know that $\langle A\rangle = \mathrm{Tr}(\rho A)$, therefore

$$\Rightarrow \langle\sigma_x\rangle = \mathrm{Tr}\left[\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix}\rho_{01} & \rho_{00}\\ \rho_{11} & \rho_{10}\end{pmatrix}$$
$$= \rho_{01} + \rho_{10} = r + r^*$$
$$\Rightarrow \langle\sigma_y\rangle = \mathrm{Tr}\left[\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix}i\rho_{01} & -i\rho_{00}\\ i\rho_{11} & -i\rho_{10}\end{pmatrix}$$
$$= i\left(\rho_{01} - \rho_{10}\right)$$
$$= i\left(r - r^*\right)$$
$$\Rightarrow \langle\sigma_z\rangle = \mathrm{Tr}\left[\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix}\rho_{00} & -\rho_{01}\\ \rho_{10} & -\rho_{11}\end{pmatrix}$$
$$= \rho_{00} - \rho_{11} = -\Delta$$

$$\Rightarrow i\langle\sigma_x\rangle + \langle\sigma_y\rangle = i\left(r + r^*\right) + i\left(r - r^*\right) = 2ir$$
$$\Rightarrow r = \frac{\langle\sigma_x\rangle}{2} - \frac{i\langle\sigma_y\rangle}{2}$$

Note that

$$i\langle\sigma_x\rangle - \langle\sigma_y\rangle = 2ir^*$$

Therefore,

$$r^* = \frac{\langle\sigma_x\rangle}{2} + \frac{i\langle\sigma_y\rangle}{2}$$

Putting these in Equation 14.43, we get

$$-\frac{\partial}{\partial t}\langle\sigma_z\rangle = -i\omega_1\left(\left(\frac{\langle\sigma_x\rangle}{2} + \frac{i\langle\sigma_y}{2}\right)e^{-i\omega t} + \left(\frac{\langle\sigma_x\rangle}{2} - i\frac{\langle\sigma_y\rangle}{2}\right)e^{i\omega t}\right) - \Gamma_1\left(-\langle\sigma_z\rangle + \langle\sigma_z\rangle^{\mathrm{eq}}\right)$$

$$\frac{\partial}{\partial t}\langle\sigma_z\rangle = \frac{i\omega_1}{2}\left(\left(\langle\sigma_x\rangle + i\langle\sigma_y\rangle\right)e^{-i\omega t} + \left(\langle\sigma_x\rangle - i\langle\sigma_y\rangle\right)e^{i\omega t}\right) - \Gamma_1\left(\langle\sigma_z\rangle - \langle\sigma_z\rangle^{\mathrm{eq}}\right)$$

Therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\langle\sigma_x\rangle - i\langle\sigma_y\rangle\right) = \left(-i\omega_0 - r_2\right)\left(\langle\sigma_x\rangle - i\langle\sigma_y\rangle\right) + \frac{i\omega_1}{2}\langle\sigma_z\rangle e^{-i\omega_t}$$

CHAPTER 17

Chapter 17

Exercise 17.1

Prove eqn. 17.1 in the special case of two pure states.

Solution: Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$. Let $|\psi^\perp\rangle$ be orthogonal state such that $|\psi\rangle$, $|\psi^\perp\rangle$ and $|\phi\rangle$ lie in the same plane in the Bloch Sphere. Since both states are pure, we can write

$$|\phi\rangle = \cos(\theta)|\psi\rangle + \sin(\theta)\left|\psi^\perp\right\rangle$$

Representing $|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$ in the $\{|\psi\rangle, |\psi^\perp\rangle\}$ basis, we have

$$\begin{aligned} |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| &= \begin{bmatrix} 1-\cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & -\sin^2(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & -\sin^2(\theta) \end{bmatrix} \end{aligned}$$

Since $\mathrm{Tr}(||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi||) = \sum_i |\lambda_i|$, and eigenvalues are invariant to the change of basis, the eigenvalues of $|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$ are

$$\begin{aligned} \lambda^2 - \sin^4(\theta) + \sin^2(\theta)\cos^2(\theta) &= 0 \\ \lambda^2 - \sin^4(\theta) + \sin^2(\theta)(1-\sin^2(\theta)) &= 0 \\ \lambda^2 - \sin^2 &= 0 \\ (\lambda - \sin(\theta))(\lambda + \sin(\theta)) &= 0 \end{aligned}$$

Therefore $\mathrm{Tr}(||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi||) = 2|\sin(\theta)|$. Note that

$$\left(\frac{1}{2}\mathrm{Tr}(||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi||)\right)^2 = 1 - \cos^2(\theta)$$

Also, based on our choice of basis, we have that

$$|\langle\phi||\psi\rangle|^2 = \cos^2(\theta)$$

Therefore,

$$\frac{1}{2}\mathrm{Tr}(||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi||) = D_q(\psi,\phi) = \sqrt{1 - |\langle\phi||\psi\rangle|^2}$$

Exercise 17.2

Give a formal proof that the optimal measurement for the trace-distance is always a PVM.

Solution: From N&Ch p.404. We are trying to prove that

$$D(\rho,\sigma) = \max_P \mathrm{Tr}(P(\rho - \sigma))$$

where P is a PVM. $\rho - \sigma$ can be expressed as $\rho - \sigma = Q - S$, where Q and S are positive operators with orthogonal support. This is because by spectral decomposition, $\rho - \sigma = UDU^\dagger$. Then, we can break down the diagonal matrix into positive an negative entries, such that

$$UDU^\dagger = U(D_+ + D_-)U^\dagger = UD_+U^\dagger - U(-D_-)U^\dagger = Q - S$$

. Note that $|\rho - \sigma| = Q + S$, since

$$|\rho - \sigma| = \sqrt{(Q-S)^\dagger(Q-S)} = \sqrt{Q^2 - 2QS + S^2} = \sqrt{Q^2 + S^2} = Q + S$$

by the orthogonal support of Q and S, so $D(\rho,\sigma) = (\mathrm{Tr}(Q) + \mathrm{Tr}(S))/2$. However, $\mathrm{Tr}(Q - S) = \mathrm{Tr}(\rho - \sigma) = 0$, so $\mathrm{Tr}(Q) = \mathrm{Tr}(S)$, and therefore $D(\rho,\sigma) = \mathrm{Tr}(Q)$. Let P be the projector onto the support of Q. Therefore,

$$\mathrm{Tr}(P(\rho - \sigma)) = \mathrm{Tr}(P(Q - S)) = \mathrm{Tr}(Q) = D(\rho,\sigma)$$

Conversely, let P be any projector. Thus

$$\mathrm{Tr}(P(\rho-\sigma)) = \mathrm{Tr}(P(Q-S)) \leq \mathrm{Tr}(PQ) \leq \mathrm{Tr}(Q) = D(\rho,\sigma)$$

which means that $D(\rho,\sigma) = \max_P \mathrm{Tr}(P(\rho-\sigma))$

Exercise 17.3

Derive Eq. 17.6 from Eq. 17.5.

Solution: Let $\rho = |\psi\rangle\langle\psi|$. Since $|\psi\rangle$ is a pure state, then $\sqrt{\rho} = \rho$. By the cyclic property of trace, we have

$$\begin{aligned}
F(|\psi\rangle,\sigma) &= \left(\mathrm{Tr}\left(\sqrt{|\psi\rangle\langle\psi|\sigma|\psi\rangle\langle\psi|}\right)\right)^2 \\
&= \left(\mathrm{Tr}\left(\sqrt{\langle\psi|\sigma|\psi\rangle|\psi\rangle\langle\psi|}\right)\right)^2 \\
&= \left(\sqrt{\langle\psi|\sigma|\psi\rangle}\,\mathrm{Tr}\left(|\psi\rangle\langle\psi|\right)\right)^2 \\
&= \left(\sqrt{\langle\psi|\sigma|\psi\rangle}\right)^2 \\
&= \langle\psi|\sigma|\psi\rangle
\end{aligned}$$