Introduction to Superoperator Representations and Characterization of Completely Positive Maps

Danesh Morales Hashemi

AMATH 900 Final Project

Waterloo, Ontario, Canada, Spring 2024

Abstract

Superoperators play a crucial role in quantum information theory, providing a framework for describing quantum channels and operations. In this project, we explore various ways to represent a superoperator, which includes the Kraus representation, Stinespring representation, Liouville matrix representation, Pauli Transfer Matrix, Choi matrix representation, and χ-matrix representation.

To begin with, we introduce the concept of superoperators, which are mathematical constructs used to describe the evolution of quantum states. We will we focus on completely positive and trace-preserving maps, which are essential in quantum information theory as they represent the most general form of physical quantum operations.

Next, we delve into different representations of superoperators. We define each of these representations in detail and prove several theorems that demonstrate their importance. These theorems highlight the utility of each representation depending on the specific properties of the superoperator we aim to investigate. For instance, the Kraus representation is particularly useful for understanding how quantum operations decompose into simpler components, while the Choi matrix provides a clear criterion for complete positivity.

Then, we prove that all the different representations of a superoperator are equivalent. This equivalence is important because it allows us to convert between representations with ease, depending on which one is more convenient for the problem at hand. That is, if we are given a certain superoperator representation, we can find all the other representations with ease.

Consequently, we analyze the properties of these representations when we add the restrictions that the superoperators are trace-preserving and/or completely positive. By imposing these restrictions, we clarify our understanding of how these properties manifest in each representation and derive conditions that must be satisfied.

Finally, we visualize specific quantum channels using these representations. This analysis provides a new perspective on how the representations depict the channels and allows us to examine their properties in greater detail.

Table of Contents

List of Figures

1 Superoperators

1.1 Introduction

Definition 1.1 (Superoperator)**.** A superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a linear map taking linear density matrices to density matrices.

Definition 1.2 (CPTP map)**.** A completely positive trace-preserving (CPTP) map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a superoperator satisfying the following conditions [1]:

(i) Convex Linearity: $p_1\mathcal{E}(\rho_1) + p_2\mathcal{E}(\rho_2) = \mathcal{E}(p_1\rho_1 + p_2\rho_2)$ where $p_i \geq 0$.

(ii) Completely positivity: For any Hilbert space $\mathcal{H}_B$ and $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have that $[\mathcal{E} \otimes \mathcal{I}_B](\rho_{AB}) \geq 0$ whenever $\rho_{AB} \geq 0$, where $\mathcal{I}_B$ is the identity superoperator acting on $\mathcal{H}_B$.

(iii) Trace preserving: $\mathrm{Tr}\big(\mathcal{E}(\rho)\big) = \mathrm{Tr}(\rho)$.

Definition 1.3 (CP map)**.** A completely positive (CP) map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a superoperator that satisfies (i) and (ii).

Definition 1.4 (Quantum Channel)**.** A quantum channel $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a CPTP map.

Definition 1.5 (Unital)**.** A superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is unital if

$$\mathcal{E}(\mathbb{1}_A) = \mathbb{1}_B \tag{1.1}$$

Remark. Even though we can have superoperators $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ have different dimensions [2], in this project we will only be focusing in superoperators of the form $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$.

1.2 Examples of Quantum Channels

Example 1.2.1 (Depolarizing Channel)*.* For a D-dimensional quantum system, the depolarizing channel can be expressed as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{\mathbb{1}}{D}p \quad \text{where} \quad 0 \leq p \leq 1 + \frac{1}{D^2 - 1} \tag{1.2}$$

Example 1.2.2 (Amplitude Damping Channel). For a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ we can represent the amplitude damping channel as

$$\mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \tag{1.3}$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{1.4}$$

and $0 \leq \gamma \leq 1$.

Example 1.2.3 (Phase Damping Channel). For a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ we can express a phase damping channel as

$$\mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \tag{1.5}$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \tag{1.6}$$

and $0 \leq \lambda \leq 1$.

Example 1.2.4 (Phase + Amplitude Damping). We can combine the phase and amplitude channels and express it as one superoperator, which is represented as

$$\mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger \tag{1.7}$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda'-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda'} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{1.8}$$

and $\lambda' = \lambda(1-\gamma)$, $0 \leq \lambda \leq 1$, $0 \leq \gamma \leq 1$ [3].

1.3 Pauli Channels

Definition 1.6 (Generalized Pauli Basis)**.** Given a n-qubit system, the generalized Pauli basis is defined as

$$\mathcal{P}_n = \{P_\alpha\} = \{\mathbb{1}, X, Y, Z\}^{\otimes n} \tag{1.9}$$

with $|\mathcal{P}_n| = 4^n = D^2$. By convention, we define the elements P_α of $\mathcal{P}_n$ as

$$\begin{aligned} P_1 &= \mathbb{1}^{\otimes n} \\ P_2 &= \mathbb{1}^{\otimes n-1} \otimes X \\ P_3 &= \mathbb{1}^{\otimes n-1} \otimes Y \\ P_4 &= \mathbb{1}^{\otimes n-1} \otimes Z \\ P_5 &= \mathbb{1}^{\otimes n-2} \otimes X \otimes \mathbb{1} \\ &\vdots \\ P_{D^2} &= Z^{\otimes n} \end{aligned}$$

Remark. Any pair of generalized Paulis will either commute or anti-commute [1].

Definition 1.7 (Pauli Channel)**.** A Pauli channel $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ of a n-qubit system is a quantum channel that can be represented as

$$\mathcal{E}(\rho) = \sum_{\alpha=1}^{4^n} p_\alpha P_\alpha \rho P_\alpha \tag{1.10}$$

where $P_\alpha \in \mathcal{P}_n$, $\alpha \in \{1, \ldots, D^2\}$ and $\sum_\alpha p_\alpha = 1$. Usually, we define $p_1 = 1 - p'$ where $p' = \sum_{\alpha \neq 1} p_\alpha$.

Recall that an arbitrary density matrix for a qubit ($n = 1$) can be written as [4]:

$$\rho = \frac{\mathbb{1}_A + \vec{r} \cdot \vec{\sigma}}{2} \tag{1.11}$$

where $\vec{r} = (r_X, r_Y, r_Z)$ is the Bloch vector, $\vec{\sigma} = (X, Y, Z)$ and $\|\vec{r}\| \leq 1$, satisfying the equality if and only if ρ represents a pure state, otherwise it's a mixed state. Therefore, using the convention definition of the generalized Pauli basis, we can rewrite ρ as

$$\rho = \frac{1}{2}\sum_{\gamma=1}^{4} r_\gamma P_\gamma \tag{1.12}$$

where $r_1 = 1$, and $(r_2, r_3, r_4) = (r_X, r_Y, r_Z)$. Expressing ρ as in (1.12) can help us see how a Pauli channel $\mathcal{E}$ acts on a density matrix ρ. We can see this by replacing (1.12) in (1.10).

$$\mathcal{E}(\rho) = \frac{1}{2}\sum_{\alpha,\gamma=1}^{4} p_\alpha r_\gamma P_\alpha P_\gamma P_\alpha \tag{1.13}$$

By the above remark, we can simplify (1.13) [5]:

$$P_\alpha P_\gamma P_\alpha = A_{\gamma,\alpha} P_\gamma, \qquad \text{with } A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{1.14}$$

Equation (1.13) will have the form

$$\mathcal{E}(\rho) = \frac{1}{2}\sum_{\gamma=1}^{4}\left(\sum_{\alpha=1}^{4} p_\alpha A_{\gamma,\alpha}\right) r_\gamma P_\gamma \tag{1.15}$$

Example 1.3.1. [Depolarizing channel is a Pauli channel] We can represent the depolarizing channel from Example 1.3.1 as a Pauli channel, with $p_1 = 1 - p + p/D^2$ and $p_\alpha = p/D^2$ for all $\alpha \neq 1$ [1]. For $D = 2$, we have

$$\mathcal{E}(\rho) = \left(1 - \frac{3}{4}p\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right) \tag{1.16}$$

2 Superoperator Representations

2.1 Kraus Representation

2.1.1 Introduction

Definition 2.1 (Kraus Operators)**.** A superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ can be described as

$$\mathcal{E}(\rho) = \sum_k A_k \rho B_k^\dagger \tag{2.1}$$

where $\{A_k\}$ and $\{B_k\}$ are sets of Kraus operators.

Proposition 2.1. CP maps can be represented as

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \tag{2.2}$$

It implies that $\{B_k\} = \{A_k\}$. This will become apparent in Section 2.4 when we learn about the Choi Representation and Choi's Theorem.

Definition 2.2 (Kraus Representation of CPTP maps)**.** $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ can be expressed as

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \tag{2.3}$$

for some set of Kraus operators $\{A_k\}$ such that $\sum_k A_k^\dagger A_k = \mathbb{1}$.

2.1.2 Kraus Representation Theorem

Theorem 2.1 (Kraus Representation Theorem [1]). A superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is a CPTP map if and only if it admits a set of Kraus operators $\{A_k\}$ and $\sum_k A_k^\dagger A_k = \mathbb{1}$.

Proof. $\implies$) Assume that $\mathcal{E}$ is a CPTP map. By Stinespring Dilation Theorem (Theorem 2.5), $\mathcal{E}$ can be expressed as

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(U\rho \otimes |0\rangle\langle 0|\, U^\dagger) \tag{2.4}$$

where $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \to \mathcal{H}_B$ is a unitary operator acting on the extended Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $\{|b_k\rangle\}$ be an orthonormal basis of $\mathcal{H}_B$. Therefore

$$\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{Tr}_B(U\rho \otimes |0\rangle\langle 0|\, U^\dagger) \\
&= \sum_k \left(\mathbb{1}_A \otimes \langle b_k|\right) U \left(\rho \otimes |0\rangle\langle 0|\right) U^\dagger \left(\mathbb{1}_A \otimes |b_k\rangle\right) \\
&= \sum_k \underbrace{\left(\mathbb{1}_A \otimes \langle b_k|\right) U \left(\mathbb{1}_A \otimes |0\rangle\right)}_{A_k} \rho \underbrace{\left(\mathbb{1}_A \otimes \langle 0|\right) U^\dagger \left(\mathbb{1}_A \otimes |b_k\rangle\right)}_{A_k^\dagger} \\
&= \sum_k A_k \rho A_k^\dagger
\end{aligned}$$

Remark. This proof assumes Stinespring's Dilation Theorem (2.5) is true, however, we used Kraus Representation Theorem to prove Theorem 2.5. To avoid this problem, Theorem 2.9 proves this implication too.

$\Longleftarrow$) Assume that $\mathcal{E}$ admits an operator-sum decomposition. Therefore

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \tag{2.5}$$

Trace Preserving: By the cyclic property of trace, we have

$$\begin{aligned}
\mathrm{Tr}(\mathcal{E}(\rho)) &= \mathrm{Tr}\left(\sum_k A_k \rho A_k^\dagger\right) \\
&= \sum_k \mathrm{Tr}\left(A_k \rho A_k^\dagger\right) \\
&= \sum_k \mathrm{Tr}\left(A_k^\dagger A_k \rho\right) \\
&= \mathrm{Tr}\left(\sum_k A_k^\dagger A_k \rho\right) \\
&= \mathrm{Tr}(\mathbb{1}\rho) \\
&= \mathrm{Tr}(\rho)
\end{aligned}$$

Complete Positivity: To show complete positivity, we need to prove that $\mathcal{I}_B \otimes \mathcal{E} \geq 0$ for any positive semidefinite $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$

Consider an extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho_{AB} \in \mathcal{L}(\rho_{AB})$ be a positive semidefinite operator. We need to show that:

$$(\mathcal{I}_B \otimes \mathcal{E})(\rho_{AB}) \geq 0 \tag{2.6}$$

Using the operator-sum decomposition:

$$(\mathcal{I}_B \otimes \mathcal{E})(\rho_{AB}) = \sum_k (A_k \otimes \mathcal{I}_B)\rho_{AB}(A_k^\dagger \otimes \mathcal{I}_B). \tag{2.7}$$

Since ρ_{AB} is positive semidefinite, we can write it as $\rho_{AB} = \sum_i |\psi_i\rangle\langle\psi_i|$ where $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.Therefore

$$(\mathcal{I}_B \otimes \mathcal{E})(\rho_{AB}) = \sum_{i,k} \underbrace{(A_k \otimes \mathcal{I}_B)|\psi_i\rangle}_{|\phi_{i,k}\rangle} \underbrace{\langle\psi_i|(A_k^\dagger \otimes \mathcal{I}_B)}_{\langle\phi_{i,k}|} = \sum_{i,k} \left|\phi_{i,k}\middle\rangle\middle\langle\phi_{i,k}\right| \geq 0 \tag{2.8}$$

finishing the proof. □

The following theorems show the different freedoms of the Kraus Representation

Theorem 2.2 (Unitary Freedom of Kraus operators). [4] Given two quantum channels $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ and $\mathcal{F} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$, then $\mathcal{E} = \mathcal{F}$ if and only if their Kraus operators are related by a unitary change of basis.

Proof. Let $\{A_k\}$ be a set of Kraus operators such that $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$. Define a new set of Kraus operators $\{B_k\}$ such that $A_k = \sum_l U_{kl} B_l$, where U is an unitary matrix. Therefore,

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_k A_k \rho A_k^\dagger \\
&= \sum_{k,l,l'} U_{kl} B_l \rho B_{l'}^\dagger U_{kl'}^* \\
&= \sum_{k,l,l'} U_{kl} U_{kl'}^* B_l \rho B_{l'}^\dagger \\
&= \sum_{l,l'} \delta_{ll'} B_l \rho B_{l'}^\dagger \\
&= \sum_l B_l \rho B_{l'}^\dagger \\
&= \mathcal{E}(\rho)
\end{aligned}$$

□

[4].

Theorem 2.3. There is a unitary freedom in choosing the set of Kraus operators associated with any fixed unitary acting on an extended Hilbert space [1].

Proof. Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ be a quantum channel such that

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(U\rho_A \otimes |0\rangle\langle 0| \, U^\dagger) = \sum_k A_k \rho_A A_k^\dagger \tag{2.9}$$

where $A_k = \langle k| \, U \, |0\rangle$. Since there is freedom assigning a state to the ancillary system, then we can choose a state $|\psi\rangle = V \, |0\rangle$ where V is a unitary acting on $|0\rangle$.

Therefore

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(U\rho_A \otimes |\psi\rangle\langle \psi| \, U^\dagger) = \sum_k B_k \rho_A B_k^\dagger \tag{2.10}$$

where $B_k = \langle k| \, U \, |\psi\rangle$. Note that

$$B_k = \langle k| \, U \, |\psi\rangle = \langle k| \, UV \, |0\rangle = \langle k| \, U' \, |0\rangle = A'_k \tag{2.11}$$

Therefore there is a unitary freedom in choosing the set of Kraus operators associated with any fixed unitary acting on an extended Hilbert space. □

Additionally, we have the following theorem.

Theorem 2.4. If $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is CPTP map then $\mathcal{E}^\dagger$ is unital [2].

Proof. Assume $\mathcal{E}$ is CPTP, hence

$$\langle \mathbb{1}, \rho\rangle = \mathrm{Tr}(\rho) = \mathrm{Tr}\big(\mathcal{E}(\rho)\big) = \langle \mathbb{1}, \mathcal{E}(\rho)\rangle = \langle \mathcal{E}^\dagger(\mathbb{1}), \rho\rangle \tag{2.12}$$

Therefore $\langle \mathbb{1} - \mathcal{E}^\dagger(\mathbb{1}), \rho\rangle = 0$, which implies that $\mathcal{E}^\dagger(\mathbb{1}) = \mathbb{1}$. This means that $\mathcal{E}$ is unital. □

Remark. All quantum channels $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ acting on a D-dimensional Hilbert space can be generated by a set of Kraus operators $\{A_k\}$ containing at most D^2 elements [4].

2.2 Stinespring Representation

2.2.1 Introduction

Definition 2.3 (Stinespring Representation [2])**.** Let $\mathcal{H}_A, \mathcal{H}_B$ be Hilbert spaces and $V, W : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$ be operators. The Stinespring representation of a superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is defined as

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(V \rho W^\dagger) \tag{2.13}$$

2.2.2 Stinespring Dilation Theorem

Theorem 2.5 (Stinespring Dilation Theorem [1]). Any CPTP map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ can be expressed as the reduced action of a unitary operator acting on an extended Hilbert space $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ as follows:

$$\mathcal{E}(\rho) = \mathrm{Tr}_B \left(U \left(|0\rangle\langle 0| \otimes \rho \right) U^\dagger \right) \tag{2.14}$$

where the initial state for the ancilla Hilbert space $|0\rangle\langle 0| \in \mathcal{L}\left(\mathcal{H}_B\right)$ is uncorrelated with the initial system state $\rho \in \mathcal{L}\left(\mathcal{H}_A\right)$.

Proof. [6] Assume $\mathcal{E}$ is CPTP. Therefore, by Theorem 2.1, $\mathcal{E}$ admits a Kraus decomposition. Let $\{A_k\}$ denote that set of Kraus operators such that

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger, \quad \sum_k A_k^\dagger A_k = \mathbb{1}_A \tag{2.15}$$

Define a unitary U in an extended Hilbert space such

$$U = \begin{pmatrix} A_0 & \boxed{?} & \cdots & \boxed{?} \\ A_1 & \boxed{?} & \cdots & \boxed{?} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N-1} & \boxed{?} & \cdots & \boxed{?} \end{pmatrix} \tag{2.16}$$

where N is the number of Kraus operators. Consider the expression

$$\mathrm{Tr}_B \left(U \left(|0\rangle\langle 0| \otimes \rho \right) U^\dagger \right) \tag{2.17}$$

Note that we can rewrite the expression as

$$\operatorname{Tr}_B\left(U\left(|0\rangle\langle 0|\otimes\rho\right)U^\dagger\right)=\operatorname{Tr}_B\left(U(|0\rangle\otimes\mathbb{1})\rho(\langle 0|\otimes\mathbb{1})U^\dagger\right)\tag{2.18}$$

From the way we chose U, we have that

$$U\left(|0\rangle\otimes\mathbb{1}\right)=\sum_{k=0}^{N-1}|k\rangle\otimes A_k\tag{2.19}$$

Therefore

$$U\left(|0\rangle\langle 0|\otimes\rho\right)U^\dagger=\sum_{j,k=0}^{N-1}|k\rangle\langle j|\otimes A_k\rho A_j^\dagger\tag{2.20}$$

and so

$$\operatorname{Tr}_B\left(U\left(|0\rangle\langle 0|\otimes\rho\right)U^\dagger\right)=\sum_{j,k=0}^{N-1}\operatorname{Tr}(|k\rangle\langle j|)A_k\rho A_j^\dagger=\sum_{k=0}^{N-1}A_k\rho A_k^\dagger=\mathcal{E}(\rho)\tag{2.21}$$

We still need to show that U is unitary. Let n be the dimension of $\mathcal{H}_A$. Therefore, the first n columns $|\gamma_n\rangle$ of U are defined as follows:

$$|\gamma_n\rangle=\sum_{k=0}^{N-1}|k\rangle\otimes A_k\,|n\rangle\tag{2.22}$$

Note that

$$\langle\gamma_a|\gamma_b\rangle=\sum_{j,k=0}^{N-1}\langle k|j\rangle\,\langle a|\,A_k^\dagger A_j\,|b\rangle=\langle a|\left(\sum_{k=0}^{N-1}A_k^\dagger A_k\right)|b\rangle=\langle a|b\rangle=\delta_{ab}\tag{2.23}$$

Therefore, the first n columns of U form an orthonormal set. We can use Gram-Schmidt to generate an orthonormal basis of $\mathcal{H}_A\otimes\mathcal{H}_B$, and thus U is a unitary matrix, finishing the proof. □

By Theorem 2.5, we can see that any CPTP can be represented as a unitary operation on an extended Hilbert space.

Remark. The dimension needed for the extended Hilbert space is going to depend on the number of Kraus operators. Since the maximum number of Kraus operators is D^2 where $D=\dim\mathcal{H}_A$, then $\max\dim\mathcal{H}_B=D^3$.

2.3 Liouville Matrix Representation

2.3.1 Introduction

Recall that a superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A)$ is a linear map. Consider the column vectorization of a $D \times D$ density matrix ρ defined as $\operatorname{vec}(\rho)$, which is a vector of length D^2. Since $\rho \mapsto \mathcal{E}(\rho)$ is linear, then the mapping

$$\operatorname{vec}(\rho) \mapsto \operatorname{vec}\big(\mathcal{E}(\rho)\big) \tag{2.24}$$

must be linear too [2]. Therefore, there exists a $D^2 \times D^2$ linear operator $[\mathcal{E}]$ that satisfies

$$[\mathcal{E}] \operatorname{vec}(\rho) = \operatorname{vec}\big(\mathcal{E}(\rho)\big) \tag{2.25}$$

Definition 2.4 (Liouville Matrix)**.** The Liouville matrix representation of a superoperator is a linear operator $[\mathcal{E}]$ that satisfies (2.25)

2.3.2 Building the Liouville Matrix

Let $\{B_\alpha\}$, $\alpha \in \{1, \ldots D^2\}$ be an orthonormal basis of the space of $D \times D$ complex matrices. The entries of the Liouville Matrix are defined as follows [1]:

$$[\mathcal{E}]_{\alpha\beta} = \operatorname{Tr}\Big(B_\alpha^\dagger \mathcal{E}(B_\beta)\Big) \tag{2.26}$$

The orthonormality of $\{B_\alpha\}$ is the defined by the Hilbert-Schmidt inner product

$$\langle B_\alpha, B_\beta \rangle = \operatorname{Tr}\Big(B_\alpha^\dagger B_\beta\Big) = \delta_{ij} \tag{2.27}$$

Remark. Even though we can choose any orthonormal basis to build the matrix, this matrix representation is unique [2], unlike the Kraus and Stinespring representation.

Example 2.3.1. By choosing the induced basis $B_\alpha = E_{ij} \equiv |i\rangle\langle j|$, the entries of the Liouville matrix are defined as

$$[\mathcal{E}]_{\alpha,\beta} = [\mathcal{E}]_{ij,kl} = \operatorname{Tr}\Big((|i\rangle\langle j|)^\dagger \mathcal{E}(|k\rangle\langle l|)\Big) = \langle i| \, \mathcal{E}(|k\rangle\langle l|) \, |j\rangle \tag{2.28}$$

The Liouville matrix in the induced basis is called Superoperator representation.

We can represent the vectorization of a density matrix ρ as a sum of projections over the vectorization of the standard basis E_α

$$\mathrm{vec}\,(\rho) = \sum_{\alpha} \mathrm{Tr}\Big(E_\alpha^\dagger \rho\Big)\,\mathrm{vec}\,(E_\alpha) \tag{2.29}$$

We can show that the definition of the Liouville Matrix satisfies Equation (2.25) [7]. From (2.29) and by the linearity and cyclic property of trace, we have

$$\begin{aligned}
\mathrm{vec}\,\big(\mathcal{E}(\rho)\big) &= \sum_{\alpha} \mathrm{Tr}\Big(E_\alpha^\dagger \mathcal{E}(\rho)\Big)\,\mathrm{vec}\,(E_\alpha) \\
&= \sum_{\alpha} \mathrm{Tr}\left(E_\alpha^\dagger \mathcal{E}\left(\sum_{\beta} \mathrm{Tr}\Big(E_\beta^\dagger \rho\Big) E_\beta\right)\right)\mathrm{vec}\,(E_\alpha) \\
&= \sum_{\alpha,\beta} \mathrm{Tr}\Big(E_\beta^\dagger \rho\Big)\,\mathrm{Tr}\Big(E_\alpha^\dagger \mathcal{E}(E_\beta)\Big)\,\mathrm{vec}\,(E_\alpha) \\
&= \sum_{\alpha,\beta} [\mathcal{E}]_{\alpha\beta}\,\mathrm{Tr}\Big(E_\beta^\dagger \rho\Big)\,\mathrm{vec}\,(E_\alpha) \\
&= \sum_{\alpha,\beta} [\mathcal{E}]_{\alpha\beta}\,\mathrm{vec}\,(\rho)_\beta\,\mathrm{vec}\,(E_\alpha) \\
&= [\mathcal{E}]\,\mathrm{vec}\,(\rho)
\end{aligned}$$

Remark. One of the advantages of the Liouville representation is that we can represent the composition of superoperators as a matrix product of their respective Liouville representations [1]. Let $\mathcal{E}_1, \mathcal{E}_2$ be a pair of superoperators of appropiate dimension. Therefore,

$$[\mathcal{E}_2][\mathcal{E}_1]\,\mathrm{vec}\,(\rho) = [\mathcal{E}_2]\,\mathrm{vec}\,\big(\mathcal{E}_1(\rho)\big) = \mathrm{vec}\,\big(\mathcal{E}_2(\mathcal{E}_1(\rho))\big) \tag{2.30}$$

2.3.3 Pauli Transfer Matrix

Definition 2.5 (Pauli Transfer Matrix)**.** The Pauli transfer matrix $[\mathcal{E}]^{\mathrm{PTM}}$ is a Liouville matrix that uses the normalized generalized Pauli basis $\{\hat{P}_\alpha\} = \{P_\alpha\}/\sqrt{D}$ as its orthonormal basis.

Remark. Definition 1.6 gives the definition of $\{P_\alpha\}$. By convention, $\hat{P}_1 = \mathbb{1}_D/\sqrt{D}$.

The entries of the Pauli transfer matrix can provide insightful information about the properties of the superoperator [1]. We can break down the Pauli transfer matrix as follows:

$$[\mathcal{E}]^{PTM} = \begin{pmatrix} t(\mathcal{E}) & \vec{m}(\mathcal{E}) \\ \vec{n}(\mathcal{E}) & [\mathbf{R}(\mathcal{E})] \end{pmatrix} \tag{2.31}$$

where t is a scalar, $\vec{m}$ and $\vec{n}$ are row and column vectors of appropiate length respectively, and $[\mathbf{R}]$ is a $D^2 - 1 \times D^2 - 1$ submatrix.

Theorem 2.6. Any trace preserving map $\mathcal{E}$ one has $t(\mathcal{E}) = 1$ and $\vec{m} = \vec{0}$, and, if $\mathcal{E}$ is a unital map, then $t(\mathcal{E}) = 1$ and $\vec{m} = \vec{n} = \vec{0}$.

Proof. Assume that $\mathcal{E}$ is a trace-preserving map. Therefore:

$$t(\mathcal{E}) = \mathcal{E}_{11} = \mathrm{Tr}(\hat{P}_1^\dagger \mathcal{E}(\hat{P}_1)) = \frac{1}{D}\,\mathrm{Tr}(\mathcal{E}(\mathbb{1}_D)) = \frac{1}{D}\,\mathrm{Tr}(\mathbb{1}_D) = 1 \tag{2.32}$$

Let $\vec{m} = (m_2, m_3, \dots m_{D^2})$ then for any $\beta \in \{2, \dots, D^2\}$

$$m_\beta = \mathcal{E}_{1\beta} = \mathrm{Tr}(\hat{P}_1^\dagger \mathcal{E}(\hat{P}_\beta)) = \frac{1}{D}\,\mathrm{Tr}(\mathbb{1}_D \mathcal{E}(\hat{P}_\beta)) = \frac{1}{D}\,\mathrm{Tr}(\hat{P}_\beta) = 0 \tag{2.33}$$

since $\mathrm{Tr}(\hat{P}_\beta) = 0$ for all Paulis except the identity matrix $\mathbb{1}_D/\sqrt{D}$. Therefore, $\vec{m} = 0$.

Assume that $\mathcal{E}$ is a unital map. Therefore, $t(\mathcal{E}) = 1$ since $\mathcal{E}(\mathbb{1}_D) = \mathbb{1}_D$.

Let $\vec{m} = (m_2, m_3, \dots m_{D^2})$ then for any $\beta \in \{2, \dots, D^2\}$

$$m_\beta = \mathrm{Tr}(\hat{P}_1^\dagger \mathcal{E}(\hat{P}_\beta)) = \frac{1}{D}\,\mathrm{Tr}(\mathbb{1}_D \mathcal{E}(\hat{P}_\beta)) = \frac{1}{D}\,\mathrm{Tr}\left(\sum_k A_k^\dagger \hat{P}_\beta A_k\right) = \frac{1}{D}\,\mathrm{Tr}(\hat{P}_\beta) = 0 \tag{2.34}$$

Moreover, for $\vec{n}$ we have

$$n_\alpha = \mathrm{Tr}(\hat{P}_\alpha^\dagger \mathcal{E}(\hat{P}_1)) = \frac{1}{D}\,\mathrm{Tr}(\hat{P}_\alpha^\dagger) = 0 \tag{2.35}$$

Therefore $\vec{m} = \vec{n} = \vec{0}$. □

Let's consider the submatrix $[\mathbf{R}]$. Recall from Equation 1.11 that we can express a density matrix for a one-qubit system as a linear combination of the Pauli matrices. By our choice of basis, then we can express $\operatorname{vec}(\rho)$ as

$$\operatorname{vec}(\rho) = \begin{pmatrix} 1 \\ r_Y \\ r_X \\ r_Z \end{pmatrix} = \begin{pmatrix} 1 \\ \vec{\mathbf{r}} \end{pmatrix} \tag{2.36}$$

Therefore, by Theorem 2.6, we have that for any trace preserving map $\mathcal{E}$

$$[\mathcal{E}] \operatorname{vec}(\rho) = \begin{pmatrix} 1 & \vec{0} \\ \vec{b} & [\mathbf{R}] \end{pmatrix} \begin{pmatrix} 1 \\ \vec{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} 1 \\ [\mathbf{R}]\vec{\mathbf{r}} + \vec{b} \end{pmatrix} = \operatorname{vec}\big(\mathcal{E}(\rho)\big) \tag{2.37}$$

We can see that $\vec{\mathbf{r}}' = [\mathbf{R}]\vec{\mathbf{r}} + \vec{b}$. If $\mathcal{E}$ is unital, then $\vec{\mathbf{r}}' = [\mathbf{R}]\vec{\mathbf{r}}$, which means that the updated components of the Bloch Sphere are just a linear transformation from the original vector. A property of $[\mathbf{R}]$ for trace-preserving maps is that their eigenvalues satisfy de inequality $|\lambda| \leq 1$ [8]. This also implies that $|\det[\mathbf{R}]| \leq 1$ since the determinant of a matrix is equal to the product of its eigenvalues. If the equality is satisfied, then $[\mathbf{R}]$ is a rotation matrix; this means that $\mathcal{E}$ maps pure states to pure states, since we are just rotating a vector that lies in the surface of the Bloch sphere. If not, then it maps pure states into mixed states.

Remark. Usually, the matrix $[\mathbf{R}]$ is diagonalizable. However, this is not always the case. A sufficient condition to ensure that $[\mathbf{R}]$ is diagonalizable is if all its eigenvalues are distinct [8].

Example 2.3.2 (One-Qubit Clifford Group). The submatrix $[\mathbf{R}]$ of the Pauli transfer matrix for the 24 elements of the one-qubit Clifford group acting on a quantum state have determinant 1 and eigenvalues satisfying $|\lambda| = 1$. It makes sense as the element of the Clifford group are unitary matrices. As an example, we have the PTMs of some elements of the Clifford group.

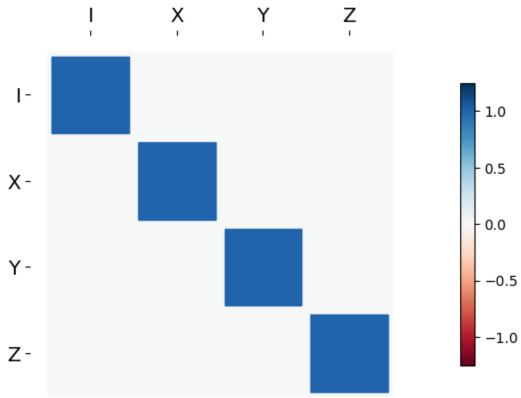

Figure 1: PTM of Pauli I gate

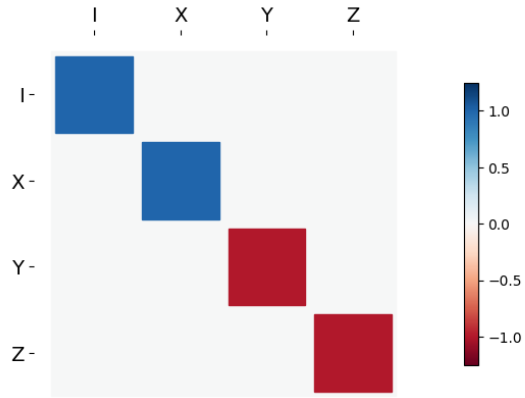

Figure 2: PTM of Pauli X gate

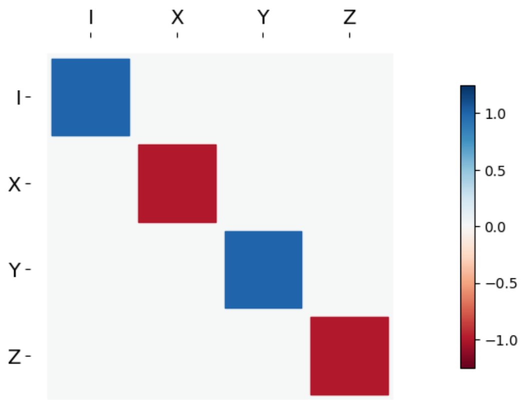

Figure 3: PTM of Pauli Y gate

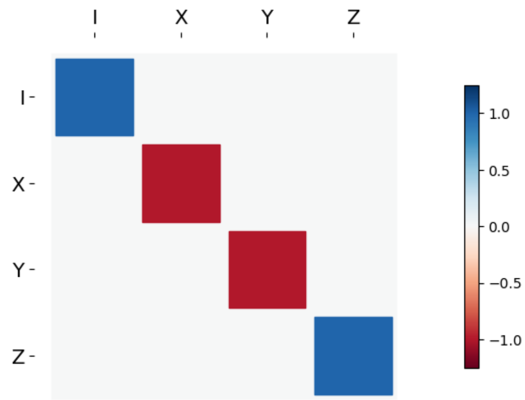

Figure 4: PTM of Pauli Z gate

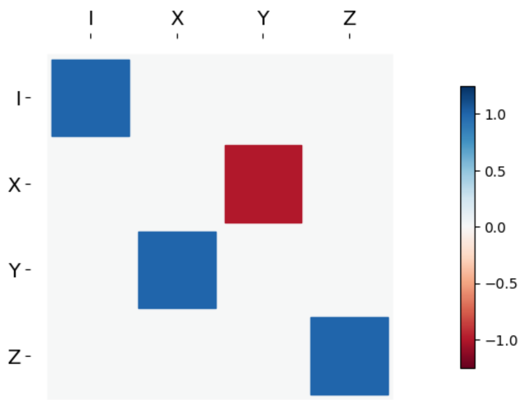

Figure 5: PTM of S gate

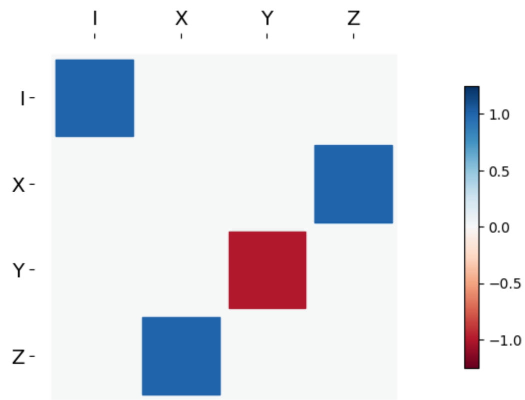

Figure 6: PTM of Hadamard gate

All the $[\mathbf{R}]$ matrices have determinant equal to 1 and satisfy $|\lambda| = 1$.

However, some channels we have seen previously do not satisfy the equalities.

Example 2.3.3 (Depolarizing Channel).

$$[\mathbf{R}] = \begin{pmatrix} 1-p & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 1-p \end{pmatrix} \tag{2.38}$$

Example 2.3.4 (Amplitude Damping Channel).

$$[\mathbf{R}] = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix} \tag{2.39}$$

Example 2.3.5 (Phase Damping Channel).

$$[\mathbf{R}] = \begin{pmatrix} \sqrt{1-\lambda} & 0 & 0 \\ 0 & \sqrt{1-\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.40}$$

Example 2.3.6 (Phase + Amplitude Damping).

$$[\mathbf{R}] = \begin{pmatrix} \sqrt{1-\lambda'-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\lambda'-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix} \tag{2.41}$$

Example 2.3.7 (One-Qubit Pauli Channel).

$$[\mathbf{R}] = \begin{pmatrix} 1-2\left(p_y+p_z\right) & 0 & 0 \\ 0 & 1-2\left(p_x+p_z\right) & 0 \\ 0 & 0 & 1-2\left(p_x+p_y\right) \end{pmatrix} \tag{2.42}$$

The eigenvalues of these matrices depend on their parameters, but we can see that only for trivial cases, the submatrices $[\mathbf{R}]$ become the identity matrix; however, for non-trivial parameter values, they are not rotation matrices.

2.4 Choi Matrix Representation

2.4.1 Introduction

Definition 2.6. (Choi Matrix) The Choi matrix $\Phi^{\mathcal{E}} : \mathcal{T}(\mathcal{H}_A, \mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_A)$ of a superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is defined as

$$\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E}) |\psi\rangle\langle\psi| = (\mathcal{I}_A \otimes \mathcal{E})(\Phi^+) \tag{2.43}$$

where $|\psi\rangle$ is the maximally entangled pure state $|\psi\rangle = \sum_i |i\rangle \otimes |i\rangle$ [9]. Expanding $|\psi\rangle$ gives us the form found in [10]

$$\begin{aligned}\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E}) |\psi\rangle\langle\psi| &= (\mathcal{I}_A \otimes \mathcal{E}) \sum_{i,j} (|i\rangle \otimes |i\rangle)(\langle j| \otimes \langle j|) \\ &= (\mathcal{I}_A \otimes \mathcal{E}) \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j| \\ &= \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)\end{aligned}$$

Therefore

$$\Phi^{\mathcal{E}} = \sum_{i,j} \mathcal{E}(E_{ij}) \otimes E_{ij} \tag{2.44}$$

The Choi matrix will take the form [1]

$$\Phi^{\mathcal{E}} = \begin{pmatrix} \mathcal{E}(E_{11}) & \mathcal{E}(E_{12}) & \dots \\ \mathcal{E}(E_{21}) & \mathcal{E}(E_{22}) & \\ \vdots & & \ddots \end{pmatrix} \tag{2.45}$$

Remark. Note that $\mathrm{vec}\,(\mathbb{1}_A) = \sum_i |i\rangle \otimes |i\rangle$. Therefore, some references such as [2] express the Choi matrix as

$$\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E})\, \mathrm{vec}\,(\mathbb{1}_A)\, \mathrm{vec}\,(\mathbb{1}_A)^\dagger \tag{2.46}$$

Proposition 2.2. The Choi matrix of a superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ with Kraus operators $\{A_k\}, \{B_k\}$ can be represented as

$$\Phi^{\mathcal{E}} = \sum_k \mathrm{vec}\,(A_k)\, \mathrm{vec}\,(B_k)^\dagger \tag{2.47}$$

Proposition 2.2 will be proved in Proposition 3.1.

2.4.2 Choi-Jamiolkowski Isomorphism

Definition 2.7 (Choi-Jamiolkowski Isomorphism)**.** There is a one to one correspondence between a superoperator $\mathcal{E}$ and its Choi matrix. We can go from $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ to $\Phi^{\mathcal{E}}$ using Equation (2.43), and we can recover $\mathcal{E}$ from $\Phi^{\mathcal{E}}$ as follows [2]:

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(\Phi^{\mathcal{E}}(\rho^T \otimes \mathbb{1}_A)) \tag{2.48}$$

To see this, we can replace Equation (2.44) into (2.48).

$$\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{Tr}_B(\Phi^{\mathcal{E}}(\rho^T \otimes \mathbb{1}_A)) \\
&= \mathrm{Tr}_B\left(\sum_{i,j}(E_{ij} \otimes \mathcal{E}(E_{ij}))(\rho^T \otimes \mathbb{1}_A)\right) \\
&= \sum_{i,j} \mathcal{E}(E_{ij}) \,\mathrm{Tr}\Big(E_{ij}\rho^T\Big) \\
&= \sum_{i,j} \rho_{ij}\mathcal{E}(E_{ij}) \\
&= \mathcal{E}\left(\sum_{i,j} \rho_{ij} E_{ij}\right) \\
&= \mathcal{E}(\rho)
\end{aligned}$$

One of the advantages of the Choi representation is that we can easily determine whether a superoperator $\mathcal{E}$ is a CP map using Choi's Theorem [10].

2.4.3 Choi's Theorem

Theorem 2.7 (Choi's Theorem)*.* A superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is a completely positive map if and only if the Choi matrix $\Phi^{\mathcal{E}}$ is positive semi-definite.

Proof. $\Longrightarrow$) Assume that $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is a completely positive map. From Equation 2.43, we have that

$$\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E}) \,|\psi\rangle\langle\psi| \tag{2.49}$$

Since $\mathcal{E}$ is a CP map and $|\psi\rangle\langle\psi|$ is positive semidefinite, then $\Phi^{\mathcal{E}}$ is positive semidefinite.

$\Longleftarrow$) Assume that $\Phi^{\mathcal{E}}$ is a positive semidefinite matrix. Therefore,

$$\Phi^{\mathcal{E}} = \sum_k |v_k\rangle\langle v_k| \tag{2.50}$$

From Proposition 2.2, let $|v_k\rangle = \mathrm{vec}\,(A_k) = \mathrm{vec}\,(B_k)$. Therefore, the superoperator $\mathcal{E}$ admits a Kraus decomposition such that

$$(\mathcal{I}_A \otimes \mathcal{E})(\rho_{AB}) = \sum_k (\mathcal{I}_A \otimes A_k)\rho_{AB}(\mathcal{I}_A \otimes A_k^\dagger) \tag{2.51}$$

Rewriting ρ_{AB} as $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ gives

$$(\mathcal{I}_A \otimes \mathcal{E})(\rho_{AB}) = \sum_{i,k} p_i(\mathcal{I}_A \otimes A_k)\,|\psi_i\rangle\langle\psi_i|\,(\mathcal{I}_A \otimes A_k^\dagger) = \sum_{i,k} p_i \left|\Psi_{i,k}\middle\rangle\middle\langle\Psi_{i,k}\right| \geq 0 \tag{2.52}$$

where $\left|\Psi_{i,k}\right\rangle = (\mathcal{I}_A \otimes A_k)\,|\psi_i\rangle$. Therefore, $\mathcal{E}$ is a CP map, finishing the proof. $\square$

Remark. Choi's Theorem (2.7) implies that a CP map admits a set of Kraus operators such that $\{A_k\} = \{B_k\}$, proving Proposition 2.1.

Theorem 2.8. A linear map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is a CPTP map if and only if $\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E})\,(\Phi^+) \geq 0$ and $\mathrm{Tr}_A\,\Phi^{\mathcal{E}} = \mathbb{1}_A$ [11].

Proof. Assume that $\mathcal{E}$ is a CPTP map. Therefore $\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E})\,(\Phi^+) \geq 0$ since $\mathcal{E}$ is CP. Moreover, note that

$$\begin{aligned}\mathrm{Tr}_A\,\Phi^{\mathcal{E}} = \mathrm{Tr}_A \sum_{i,j} E_{i,j} \otimes \mathcal{E}(E_{i,j}) &= \sum_{i,j} \mathrm{Tr}(\mathcal{E}(E_{i,j}))E_{i,j} \\ &= \sum_{i,j} \delta_{i,j} E_{i,j} \\ &= \sum_i E_{i,i} = \mathbb{1}_A\end{aligned}$$

Now, assume that $\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E})(\Phi^+) \geq 0$ and $\operatorname{Tr}_A \Phi^{\mathcal{E}} = \mathbb{1}_A$. Then $\mathcal{E}$ is a trace preserving map, since

$$\begin{aligned}
\operatorname{Tr} \mathcal{E}(\rho) = \operatorname{Tr}_A \left(\operatorname{Tr}_B \left[\Phi^{\mathcal{E}} (\rho^T \otimes \mathbb{1}_A) \right] \right) &= \operatorname{Tr}_B \left(\operatorname{Tr}_A \left[\Phi^{\mathcal{E}} (\rho^T \otimes \mathbb{1}_A) \right] \right) \\
&= \operatorname{Tr}_B \left(\operatorname{Tr}_A \left[\Phi^{\mathcal{E}} (\mathbb{1}_B \otimes \mathbb{1}_A) \right] \rho^T \right) \\
&= \operatorname{Tr} \rho^T = \operatorname{Tr} \rho
\end{aligned}$$

We need to show that $(\mathcal{I}_B \otimes \mathcal{E})(\rho_{AB}) \geq 0$. Note that we can rewrite positive semidefinite operators, such as ρ_{AB}, in the form

$$\rho_{AB} = \sum_k |\psi_k\rangle\langle\psi_k| \tag{2.53}$$

where $|\psi_k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Furthermore, each $|\psi_k\rangle$ can be written as

$$|\psi_k\rangle = \sum_i \sum_j A_{ij}^{(k)} |i\rangle \otimes |j\rangle = (A_k \otimes \mathbb{1}_A) \sum_i |i\rangle \otimes |i\rangle = (A_k \otimes \mathbb{1}_A) \left|\phi^+\right\rangle \tag{2.54}$$

Therefore,

$$\begin{aligned}
(\mathcal{I}_B \otimes \mathcal{E})(\rho_{AB}) &= (\mathcal{I}_B \otimes \mathcal{E}) \left(\sum_k (A_k \otimes \mathbb{1}_A) \left|\phi^+\middle\rangle\middle\langle\phi^+\right| \left(A_k \otimes \mathbb{1}_A^\dagger\right) \right) \\
&= \sum_k (A_k \otimes \mathbb{1}_A) (\mathcal{I}_B \otimes \mathcal{E})(\Phi^+) \left(A_k \otimes \mathbb{1}_A^\dagger\right) \\
&= \sum_k (A_k \otimes \mathbb{1}_A) \Phi^{\mathcal{E}} \left(A_k \otimes \mathbb{1}_A^\dagger\right) \\
&= \sum_k \mathbf{A}_k \Phi^{\mathcal{E}} \mathbf{A}_k^\dagger \geq 0
\end{aligned}$$

since $\Phi^{\mathcal{E}} \geq 0$, finishing the proof. □

Theorem 2.9. Let $\mathcal{E}:\mathcal{L}(\mathcal{H}_A)\to\mathcal{L}(\mathcal{H}_A)$ be a CPTP map. It is always possible to find a set of Kraus operators for $\mathcal{E}$ [11].

Proof. Recall that a linear map $\mathcal{E}$ is CPTP if and only if $\Phi^{\mathcal{E}} = (\mathcal{I}_A\otimes\mathcal{E})(\Phi^+)\geq 0$ and $\operatorname{Tr}_A J(\mathcal{E}) = \mathbb{1}_A$. By the Choi-Jamiolkowski Isomorphism, given a operator $\Phi^{\mathcal{E}}$, we can also define a mapping by

$$\mathcal{E}(\rho) = \operatorname{Tr}_B\left[\Phi^{\mathcal{E}}\left(\rho^T\otimes\mathbb{1}_A\right)\right] \tag{2.55}$$

Let $\Phi^{\mathcal{E}}$ be an arbitrary positive semi-definite operator. Therefore, it can be decomposed as

$$\Phi^{\mathcal{E}} = \sum_k |\psi_k\rangle\langle\psi_k| \tag{2.56}$$

for some collection of eigenvectors $|\psi_k\rangle$. Furthermore, each $|\psi_k\rangle$ can be written as

$$|\psi_k\rangle = \sum_i\sum_j A_{ij}^{(k)}|i\rangle\otimes|j\rangle = (A_k\otimes\mathbb{1}_B)\sum_j|j\rangle\otimes|j\rangle = (A_k\otimes\mathbb{1}_B)\left|\phi^+\right\rangle \tag{2.57}$$

where each A_k is square matrix with entries $A_{i,j}^{(k)}$. Therefore

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_j \operatorname{Tr}_B\left[|\psi_k\rangle\langle\psi_k|\left(\rho^T\ \otimes\mathbb{1}_A\right)\right]\\
&= \sum_k \operatorname{Tr}_B\left[(A_k\otimes\mathbb{1}_B)\left|\phi^+\right\rangle\left\langle\phi^+\right|(A_k^*\otimes\mathbb{1}_B)\left(\rho^T\otimes\mathbb{1}_A\right)\right]\\
&= \sum_k A_k\left(\operatorname{Tr}_B\left[\left|\phi^+\right\rangle\left\langle\phi^+\right|\left(\rho^T\otimes\mathbb{1}_A\right)\right]\right)A_k^\dagger\\
&= \sum_k A_k\left(\operatorname{Tr}_B\left[\left|\phi^+\right\rangle\left\langle\phi^+\right|(\mathbb{1}_A\otimes\rho)\right]\right)A_k^\dagger\\
&= \sum_k A_k\left(\underbrace{\operatorname{Tr}_B\left[\left|\phi^+\right\rangle\left\langle\phi^+\right|\right]}_{\mathbb{1}}\right)\rho A_k^\dagger\\
&= \sum_k A_k\rho A_k^\dagger
\end{aligned}$$

Moreover, note that

$$
\begin{aligned}
\sum_k A_k^\dagger A_k &= \sum_k A_k^\dagger \mathbb{1} A_k \\
&= \sum_k A_k^\dagger \operatorname{Tr}_A \left[\left| \phi^+ \middle\rangle\middle\langle \phi^+ \right| \right] A_k \\
&= \sum_k \operatorname{Tr}_A \left[\left(A_k^\dagger \otimes \mathbb{1}_B \right) \left| \phi^+ \middle\rangle\middle\langle \phi^+ \right| \left(A_k \otimes \mathbb{1}_B \right) \right] \\
&= \overline{\sum_k \operatorname{Tr}_A \left[\left(A_k \otimes \mathbb{1}_B \right) |\phi^+\rangle\langle\phi^+| \left(A_k^\dagger \otimes \mathbb{1}_B \right) \right]} \\
&= \overline{\sum_k \operatorname{Tr}_A \left| \psi_j \middle\rangle\middle\langle \psi_j \right|} \\
&= \overline{\operatorname{Tr}_A \Phi^{\mathcal{E}}} \\
&= \bar{\mathbb{1}} = \mathbb{1}
\end{aligned}
$$

Therefore, any CPTP map admits an operator-sum decomposition. □

Corollary 2.1. Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ be a nonzero linear operator, and let $r = \operatorname{rank} \Phi^{\mathcal{E}}$. There exists a Kraus representation

$$
\mathcal{E}(\rho) = \sum_{k=0}^{r-1} A_k \rho B_k^\dagger \tag{2.58}
$$

for some choice of linear operators $\{A_k\}, \{B_k\} \subset \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ [2].

Proof. Since $r = \operatorname{rank} \Phi^{\mathcal{E}}$, then

$$
\Phi^{\mathcal{E}} = \sum_{k=0}^{r-1} |u_k\rangle\langle v_k| \tag{2.59}
$$

for some set of vectors $\{|u_k\rangle\}, \{|v_k\rangle\} \subset \mathcal{H}_A \otimes \mathcal{H}_A$. Choose $|u_k\rangle = \operatorname{vec}(A_k)$ and $|v_k\rangle = \operatorname{vec}(B_k)$. Therefore, it follows from Proposition 2.2 that Equation 2.58 is the Kraus Representation of $\mathcal{E}$. □

Corollary 2.2. Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ be a nonzero linear operator, and let $r = \operatorname{rank} \Phi^{\mathcal{E}}$. For an ancillary space $\mathcal{H}_B$ with $r = \dim \mathcal{H}_B$, there exists a Stinespring representation of $\mathcal{E}$ of the form

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(A\rho B^\dagger) \tag{2.60}$$

for some choice of linear operators $V, W : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$ [2].

Proof. Let $\dim \mathcal{H}_B = r$. We can define

$$A = \sum_k^r A_k \otimes |k\rangle\,, \quad B = \sum_k^r B_k \otimes |k\rangle\,. \tag{2.61}$$

where $\{A_k\}, \{B_k\}$ are the Kraus operators of $\mathcal{E}$, and $\{|k\rangle\}$ is an orthornormal basis of the ancillary Hilbert space $\mathcal{H}_B$ with dimension r it holds that Equation 2.60 is the Stinespring representation of $\mathcal{E}$ with Kraus operators $\{A_k\}, \{B_k\}$. □

2.5 $\chi-$Matrix Representation

Definition 2.8 ($\chi -$ Matrix)**.** Given a basis $\{B_\alpha\}$ of $\mathcal{H}_A$, we can represent any superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ as

$$\mathcal{E}(\rho) = \sum_{\alpha,\beta} \chi_{\alpha\beta} B_\alpha \rho B_\beta^\dagger \tag{2.62}$$

where $\chi_{\alpha\beta}$ are the entries of the χ-Matrix [1].

Remark. While any orthonormal basis can be chosen to construct the χ-Matrix, the generalized Pauli basis $\mathcal{P}_n$ is typically preferred and will be used in this project [12].

We can construct the $\chi-$Matrix from the Kraus representation of $\mathcal{E}$. Given $A_k = \sum_\alpha a_{k\alpha} P_\alpha$, $B_k = \sum_\beta b_{k\beta} P_\beta$, where $P_\alpha, P_\beta \in \mathcal{P}_n$, then

$$\mathcal{E}(\rho) = \sum_{\alpha,\beta} \chi_{\alpha\beta} P_\alpha \rho P_\beta^\dagger, \quad \text{where} \quad \chi_{\alpha\beta} = \sum_k a_{k\alpha} b_{k\beta}^* \tag{2.63}$$

Corollary 2.3. The $\chi-$Matrix of a CP map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is Hermitian and positive semidefinite.

Proof. Assume that $\mathcal{E}$ is a CP map, therefore $\{B_k\} = \{A_k\}$ and the entries of the $\chi-$Matrix are given by

$$\chi_{\alpha\beta} = \sum_k a_{k\alpha} a^*_{k\beta} = \chi^*_{\beta\alpha} \tag{2.64}$$

Therefore, χ is Hermitian. Moreover,

$$\begin{aligned}
\langle x| \chi |x\rangle &= \sum_{\alpha,\beta} x^*_\alpha \chi_{\alpha\beta} x_\beta \\
&= \sum_{\alpha,\beta,k} (a_{k\alpha} x^*_\alpha)(a^*_{k\beta} x_\beta) \\
&= \sum_k \left| \sum_\beta a_{k\beta} x^*_\beta \right|^2 \geq 0
\end{aligned}$$

Which implies that χ is positive semidefinite. □

Example 2.5.1 (χ-Matrix of Pauli Channel.). From the definition of a Pauli Channel (1.7) and $\chi-$Matrix (2.8), we can see that

$$\mathcal{E}(\rho) = \sum_\alpha p_\alpha P_\alpha \rho P_\alpha = \sum_{\alpha,\beta} p_\alpha \delta_{\alpha\beta} P_\alpha \rho P_\beta = \sum_{\alpha,\beta} \chi_{\alpha\beta} P_\alpha \rho P_\beta \tag{2.65}$$

Therefore,

$$\chi_{\alpha\beta} = p_\alpha \delta_{\alpha\beta} \tag{2.66}$$

This means that the $\chi-$Matrix of a Pauli channel is a diagonal matrix, where the diagonal entries are given by the Pauli probabilities as follows:

$$\chi = \text{diag}(p_1, \ldots, p_{4^n}) = \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & p_{4^n} \end{pmatrix} \tag{2.67}$$

Therefore, for a Pauli channel $\mathcal{E}$, $\text{Tr}(\chi) = 1$. We will see in Example 3.2.2 that if a superoperator is trace-preserving, then $\text{Tr}(\chi) = 1$.

3 Equivalence of Superoperator Representations

3.1 Introduction

In this section, we will show the relationship among the different superoperator representations and prove them. The following proposition is from [2], however we have added the $\chi-$Matrix representation.

3.2 Relationship Among Representations

Proposition 3.1. Let $\mathcal{H}_A$ be a Hilbert space, let $\{A_k\}, \{B_k\}$ be sets of Kraus operators, and let $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ be a linear superoperator. The following are equivalent:

1. **Kraus Representation:**
$$\mathcal{E}(\rho) = \sum_k A_k \rho B_k^\dagger \tag{3.1}$$

2. **Superoperator Representation:**
$$[\mathcal{E}] = \sum_k A_k \otimes \bar{B}_k \tag{3.2}$$

3. **Stinespring Representation:** For $\mathcal{H}_B = \mathbb{C}^n$ and $A, B : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$
$$A = \sum_k^n A_k \otimes |k\rangle\,, \quad B = \sum_k^n B_k \otimes |k\rangle \tag{3.3}$$
we have
$$\mathcal{E}(\rho) = \mathrm{Tr}_B(A\rho B^\dagger) \tag{3.4}$$

4. **Choi Representation:**
$$\Phi^{\mathcal{E}} = \sum_k \mathrm{vec}\,(A_k)\,\mathrm{vec}\,(B_k)^\dagger \tag{3.5}$$

5. $\chi-$**Matrix Representation:** Given $A_k = \sum_\alpha a_{k\alpha} P_\alpha$, $B_k = \sum_\beta b_{k\beta} P_\beta$ then
$$\mathcal{E}(\rho) = \sum_{\alpha,\beta} \chi_{\alpha,\beta} P_\alpha \rho P_\beta^\dagger, \quad \text{where} \quad \chi_{\alpha\beta} = \sum_k a_{k\alpha} b_{k\beta}^* \tag{3.6}$$

Proof. (1) Kraus $\iff$ (2) Superoperator.

$$\begin{aligned}
\mathrm{vec}\left(\mathcal{E}(\rho)\right) = \mathrm{vec}\left(\sum_k A_k \rho B_k^\dagger\right) &= \mathrm{vec}\left(\sum_k A_k \rho (B_k^*)^T\right) \\
&= \sum_k A_k \otimes B_k^* \,\mathrm{vec}\,(\rho) \\
&= [\mathcal{E}]\,\mathrm{vec}\,(\rho)
\end{aligned}$$

which follows from Roth's Lemma.

(1) Kraus $\iff$ (3) Stinespring.

$$\begin{aligned}
\mathcal{E}(\rho) = \mathrm{Tr}_B(A\rho B^\dagger) &= \mathrm{Tr}_B\left(\sum_{k,k'} (A_k \otimes |k\rangle)\rho(B_{k'}^\dagger \otimes \langle k'|)\right) \\
&= \mathrm{Tr}_B\left(\sum_{k,k'} A_k \rho B_{k'}^\dagger \otimes |k\rangle\langle k'|\right) \\
&= \sum_{k,k'} A_k \rho B_{k'}^\dagger \otimes \mathrm{Tr}\left(|k\rangle\langle k'|\right) \\
&= \sum_{k,k'} A_k \rho B_{k'}^\dagger \delta_{kk'} \\
&= \sum_k A_k \rho B_k^\dagger
\end{aligned}$$

(1) Kraus $\iff$ (4) Choi. From Equation 2.44, we have that

$$\begin{aligned}
\Phi^{\mathcal{E}} = \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|) &= \sum_{i,j,k} A_k \,|i\rangle\langle j|\, B_k^\dagger \otimes |i\rangle\langle j| \\
&= \sum_{i,j,k} (\mathbb{1} \otimes A_k)(|i\rangle\langle j| \otimes |i\rangle\langle j|)(\mathbb{1} \otimes B_k^\dagger) \\
&= \sum_k ((\mathbb{1} \otimes A_k)\,\mathrm{vec}\,(\mathbb{1}))(\mathrm{vec}\,(\mathbb{1})^\dagger\,(\mathbb{1} \otimes B_k^\dagger)) \\
&= \sum_k ((\mathbb{1} \otimes A_k)\,\mathrm{vec}\,(\mathbb{1}))((\mathbb{1} \otimes B_k)\,\mathrm{vec}\,(\mathbb{1}))^\dagger \\
&= \sum_k \mathrm{vec}\,(A_k)\,\mathrm{vec}\,(B_k)^\dagger
\end{aligned}$$

(1) Kraus $\iff$ (5) χ-Matrix

$$\begin{aligned}\mathcal{E}(\rho) &= \sum_k A_k \rho B_k^\dagger \\ &= \sum_{\alpha,\beta,k} (a_{k\alpha} P_\alpha)\rho(b_{k\beta}P_\beta)^\dagger \\ &= \sum_{\alpha,\beta,k} a_{k\alpha} b_{k\beta}^* P_\alpha \rho P_\beta^\dagger \\ &= \sum_{\alpha,\beta} \chi_{\alpha\beta} P_\alpha \rho P_\beta^\dagger\end{aligned}$$

□

We have demonstrated that it is possible to transition from the Kraus representation to any other representation, and vice-versa. Consequently, one can convert between any two representations by first translating to the Kraus representation and then to the desired form.

However, it is not necessary to always use the Kraus representation as an intermediary. In fact, there are many interesting relationships between the different representations.

Example 3.2.1. The Superoperator and Choi Matrix contain identical elements, but their entries have been rearranged into different positions by the following reshuffling operation [13]:

$$[\mathcal{E}]_{\alpha,\beta} = [\mathcal{E}]_{ij,mn} = \Phi^{\mathcal{E}}_{nj,mi} \tag{3.7}$$

To visualize the permutations of entries, consider the following matrix representations for a 2-dimensional Hilbert space.

$$[\mathcal{E}] = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \Longrightarrow \begin{pmatrix} a & i & c & k \\ e & m & g & o \\ b & j & d & l \\ f & n & h & p \end{pmatrix} = \Phi^{\mathcal{E}} \tag{3.8}$$

In Section 5, we will observe the different representations of several well-known quantum channels. It will be evident that the relation between the Superoperator and Choi matrix is given by Equation 3.8.

Example 3.2.2. The Pauli Transfer Matrix (PTM) and the $\chi-$Matrix both use the generalized Pauli Basis. Therefore, we can have a relationship between the entries of the PTM and $\chi-$Matrix. For a $d-$dimensional Hilbert space, and the definition of the entries of a PTM and the representation of a superoperator $\mathcal{E}$ in terms of the $\chi-$Matrix, we have

$$[\mathcal{E}]_{ij}^{\mathrm{PTM}} = \mathrm{Tr}\Big(\hat{P}_i \mathcal{E}(\hat{P}_j)\Big) = \frac{1}{d}\,\mathrm{Tr}\left(P_i \sum_{k,l} \chi_{kl} P_k P_j P_l\right) = \frac{1}{d}\sum_{k,l} \chi_{kl}\,\mathrm{Tr}\big(P_i P_k P_j P_l\big) \tag{3.9}$$

By the properties of Pauli matrices, the trace of the product of four Pauli matrices in a $d-$dimensional Hilbert space can have five outcomes: $0, \pm d, \pm id$. The outcomes will depend in the selection of the four Paulis and their order, up to a cyclic permutation. This means that we can express the entries of the PTM as a linear combination of the entries of the $\chi-$Matrix, where the coefficients are $0, \pm 1$, and $\pm i$.

$$[\mathcal{E}]_{ij}^{\mathrm{PTM}} = \sum_{k,l} \chi_{kl}\zeta_{ijkl}, \quad \zeta_{ijkl} \in \{0, 1, -1, i, -1\} \tag{3.10}$$

For simplicity, consider the first entry of the PTM, $[\mathcal{E}]_{11}^{\mathrm{PTM}}$.

$$[\mathcal{E}]_{11}^{\mathrm{PTM}} = \frac{1}{d}\sum_{k,l} \chi_{kl}\,\mathrm{Tr}(P_k P_l) = \sum_k \chi_{kk} = \mathrm{Tr}(\chi) \tag{3.11}$$

This means that the first entry of a PTM is equal to the sum of the diagonal entries of the $\chi-$Matrix! Combined with Theorem 2.6, we have the following lemma.

Lemma 3.3. If $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ is a trace-preserving map or unital, then $\mathrm{Tr}(\chi) = 1$.

Example 3.3.1. The PTM matrix $[\mathcal{E}]^{\mathrm{PTM}}$ and Superoperator matrix $[\mathcal{E}]$ are built the same way, but they have different bases. Consequently, they are related by a change of basis operator [14]. For a unitary U defined as

$$U = \sum_k |k\rangle\,\mathrm{vec}\,(P_k)^\dagger \tag{3.12}$$

Then

$$[\mathcal{E}]^{\mathrm{PTM}} = U[\mathcal{E}]U^\dagger, \quad [\mathcal{E}] = U^\dagger[\mathcal{E}]^{\mathrm{PTM}}U \tag{3.13}$$

4 Characterization of CP and CPTP maps

We have just analyzed the case of a general linear superoperator. However, we can further examine how the equivalence is affected by imposing constraints on our superoperator. For example, what happens when $\mathcal{E}$ is a CP map?

From Proposition 2.1, this implies that $\{B_k\} = \{A_k\}$. Therefore we can particularize Proposition 3.1 by replacing $\{B_k\}$ with $\{A_k\}$. Applying this restriction gives us new information about the characterization of CP maps. For example, the Choi matrix becomes

$$\Phi^{\mathcal{E}} = \sum_k \mathrm{vec}\left(A_k\right) \mathrm{vec}\left(A_k\right)^\dagger \geq 0 \tag{4.1}$$

which follows from Choi's Theorem (2.7). We will now take Proposition 3.1 and several other definitions, theorems, corollaries from this project, and construct a rigorous characterization of completely positive maps.

Theorem 4.1 (Characterizations of CP maps [2]). Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ be a nonzero superoperator. The following are equivalent:

1. $\mathcal{E}$ is a CP map.

2. $\mathcal{E} \otimes \mathcal{I}_A$ is positive.

3. $\Phi^{\mathcal{E}} \geq 0$

4. Kraus Representation: There exists a set of Kraus operators $\{A_k\}$ such that

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \tag{4.2}$$

5. (4) holds for a set $\{A_k\}$ with $|\{A_k\}| = \mathrm{rank}\, \Phi^{\mathcal{E}}$.

6. Superoperator Representation:

$$[\mathcal{E}] = \sum_k A_k \otimes A_k^* \tag{4.3}$$

7. Stinespring Representation: For $\mathcal{H}_B = \mathbb{C}^n$ and $A : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$

$$A = \sum_k^n A_k \otimes |k\rangle \tag{4.4}$$

we have

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(A\rho A^\dagger) \tag{4.5}$$

8. (7) holds for an ancillary system of dimension $\operatorname{rank} \Phi^{\mathcal{E}}$.

9. Choi Representation:

$$\Phi^{\mathcal{E}} = \sum_k \operatorname{vec}(A_k) \operatorname{vec}(A_k)^\dagger \tag{4.6}$$

10. $\chi-$Matrix Representation: Given $A_k = \sum_\alpha a_{k\alpha} P_\alpha$, then

$$\mathcal{E}(\rho) = \sum_{\alpha,\beta} \chi_{\alpha,\beta} P_\alpha \rho P_\beta^\dagger, \quad \text{where} \quad \chi_{\alpha\beta} = \sum_k a_{k\alpha} a_{k\beta}^* \tag{4.7}$$

Proof. We will prove the theorem by the following sequence of implications.

(1) $\implies$ (2). It follows trivially from the definition of a CP map (1.3).

(2) $\implies$ (3). Since $\mathcal{E} \otimes \mathcal{I}_A$ is positive, then from Equation 2.46

$$\Phi^{\mathcal{E}} = (\mathcal{I}_A \otimes \mathcal{E})(\operatorname{vec}(\mathbb{1}) \operatorname{vec}(\mathbb{1})^\dagger) \tag{4.8}$$

is positive semidefinite since $\operatorname{vec}(\mathbb{1}) \operatorname{vec}(\mathbb{1})^\dagger$ is positive semidefinite.

(3) $\implies$ (5). It follows from Choi's Theorem, Proposition 2.2 and Corollary 2.1.

(5) $\implies$ (4). It follows trivially from Corollary 2.1.

(4) $\implies$ (1). It follows from Proposition 2.1.

(5) $\implies$ (8). It follows from Corollary 2.1 and 2.2.

(8) $\implies$ (7). It follows trivially from Corollary 2.2.

(7) $\implies$ (1). It follows from Proposition 2.1 and 3.1.

(1) $\iff$ (6) $\iff$ (9) $\iff$ (10). It follows from Proposition 2.1 and 3.1. □

$$
\begin{tikzcd}
 & 6 \arrow[d, leftrightarrow] & & 4 \arrow[dll] & \\
9 \arrow[r, leftrightarrow] & 1 \arrow[r] \arrow[d, leftrightarrow] & 2 \arrow[r] & 3 \arrow[r] & 5 \arrow[ul] \arrow[d] \\
 & 10 & & 7 \arrow[ull] & 8 \arrow[l]
\end{tikzcd}
$$

Figure 7: Diagram of the implications to prove the equivalence of Theorem 4.1

Similar to Theorem 4.1, we can add the condition of trace preservation and analyze the property of the representations of CPTP maps.

Theorem 4.2 (Characterizations of CPTP maps [2]). Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ be a superoperator. The following statements are equivalent:

1. $\mathcal{E}$ is a CPTP map.

2. $\Phi^{\mathcal{E}} \geq 0$ and $\mathrm{Tr}_B(\Phi^{\mathcal{E}}) = \mathbb{1}_A$

3. There exists a set of Kraus operators $\{A_k\}$ such that

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \tag{4.9}$$

   that satisfies $\sum_k A_k^\dagger A_k = \mathbb{1}$

4. (3) holds for a set $\{A_k\}$ with $|\{A_k\}| = \mathrm{rank}\, \Phi^{\mathcal{E}}$.

5. For $\mathcal{H}_B = \mathbb{C}^n$ there exists an isometry $A : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$A = \sum_k^n A_k \otimes |k\rangle \tag{4.10}$$

   we have

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(A \rho A^\dagger) \tag{4.11}$$

6. (5) holds for an ancillary system of dimension $\mathrm{rank}\, \Phi^{\mathcal{E}}$.

7. $\mathcal{E}$ can be expressed as the reduced action of a unitary operator acting on an extended Hilbert space $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ as follows:

$$\mathcal{E}(\rho) = \mathrm{Tr}_B \left(U\rho \otimes |0\rangle\langle 0|\, U^\dagger \right), \tag{4.12}$$

where the initial state for the ancilla Hilbert space $|0\rangle\langle 0|$ is uncorrelated with the initial system state ρ.

Proof. We will prove the theorem by the following sequence of implications.

(1) $\iff$ (2). It follows from Theorem 2.8.

(1) $\iff$ (3). It follows from Kraus Representation Theorem (2.1).

(2) $\implies$ (4) It follows from Choi's Theorem, Proposition 2.2 and Corollary 2.1.

(4) $\implies$ (3) It follows trivially from Corollary 2.1.

(3) $\implies$ (5). It follows from Theorem 4.1, and the fact that A is an isometry, since

$$A^\dagger A = \sum_{k,k'} A_k^\dagger A'_k \otimes \langle k | k' \rangle = \sum_{k,k'} A_k^\dagger A'_k \delta_{kk'} = \sum_k A_k^\dagger A_k = \mathbb{1}_A \tag{4.13}$$

(4) $\implies$ (6). It follows from Corollary 2.1 and 2.2.

(6) $\implies$ (5). It follows trivially from Corollary 2.2.

(5) $\implies$ (7). From the proof of Stinespring Dilation Theorem (2.5), we can choose an unitary U in the extended Hilbert space such that $U(\mathbb{1} \otimes |0\rangle) = \sum_k A_k \otimes |k\rangle = A$.

(7) $\implies$ (1). It follows from the Stinespring Dilation Theorem (2.5). $\square$

$$\begin{tikzcd}
2 \arrow[r] \arrow[d, leftrightarrow] & 4 \arrow[r] \arrow[d] & 6 \arrow[d] \\
1 \arrow[r, leftrightarrow] & 3 \arrow[r] & 5 \arrow[d] \\
 & & 7 \arrow[ull]
\end{tikzcd}$$

Figure 8: Diagram of the implications to prove the equivalence of Theorem 4.2

5 Examples of Quantum Channel Representations

5.1 Amplitude Damping Channel

Kraus Representation: As seen in Example 1.2.2, the Kraus operators are

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{5.1}$$

Stinespring Representation: $\mathcal{E}(\rho) = \mathrm{Tr}_B(V\rho V^\dagger)$, where V is defined as follows

$$V = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{5.2}$$

Superoperator Representation: The matrix $[\mathcal{E}]$ such that $[\mathcal{E}]\,\mathrm{vec}\,(\rho) = \mathrm{vec}\,\big(\mathcal{E}(\rho)\big)$ in the induced basis $\{E_\alpha\}$ is represented as

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix} \tag{5.3}$$

Pauli Transfer Matrix: The matrix $[\mathcal{E}]$ such that $[\mathcal{E}]\,\mathrm{vec}\,(\rho) = \mathrm{vec}\,\big(\mathcal{E}(\rho)\big)$ in the normalized Pauli basis $\{P_\alpha\}$ is represented as

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ \gamma & 0 & 0 & 1-\gamma \end{pmatrix} \tag{5.4}$$

Choi Representation: The Choi matrix $\Phi^{\mathcal{E}}$ such that $\mathcal{E}(\rho) = \mathrm{Tr}_A(\Phi^{\mathcal{E}}(\mathbb{1}_A \otimes \rho^T))$ is

$$\Phi^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix} \tag{5.5}$$

χ**-Matrix**: The χ-matrix in the normalized Pauli basis $\{P_\alpha\}$ is

$$\chi = \frac{1}{4} \begin{pmatrix} (1+\sqrt{1-\gamma})^2 & 0 & 0 & \gamma \\ 0 & \gamma & -i\gamma & 0 \\ 0 & i\gamma & \gamma & 0 \\ \gamma & 0 & 0 & (1-\sqrt{1-\gamma})^2 \end{pmatrix} \tag{5.6}$$

5.2 Phase Damping Channel

Kraus Representation: From Example 1.2.3:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \tag{5.7}$$

Stinespring Representation:

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{1-\lambda} \\ 0 & \sqrt{\lambda} \end{pmatrix} \tag{5.8}$$

Superoperator Representation:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & \lambda \\ 0 & \sqrt{1-\lambda} & 0 & 0 \\ 0 & 0 & \sqrt{1-\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.9}$$

Pauli Transfer Matrix:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda} & 0 & 0 \\ 0 & 0 & \sqrt{1-\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.10}$$

Choi Representation:

$$\Phi^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\lambda} & 0 & 0 & 1 \end{pmatrix} \tag{5.11}$$

χ**-Matrix**:

$$\chi = \frac{1}{2} \begin{pmatrix} 1+\sqrt{1-\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sqrt{1-\lambda} \end{pmatrix} \tag{5.12}$$

5.3 Amplitude + Phase Damping Channel

Kraus Representation: From Example 1.2.4:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda'-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda'} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{5.13}$$

Stinespring Representation:

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{\gamma} \\ 0 & \sqrt{1-\lambda'-\gamma} \\ 0 & \sqrt{\lambda'} \\ 0 & 0 \end{pmatrix} \tag{5.14}$$

Superoperator Representation:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & \sqrt{1-\lambda'-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\lambda'-\gamma} & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix} \tag{5.15}$$

Pauli Transfer Matrix:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda'-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\lambda'-\gamma} & 0 \\ \gamma & 0 & 0 & 1-\gamma \end{pmatrix} \tag{5.16}$$

Choi Representation:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\lambda'-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\lambda'-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix} \tag{5.17}$$

χ-Matrix Representation:

$$\chi = \frac{1}{4}\begin{pmatrix} \left(1+\sqrt{1-\lambda'-\gamma}\right)^2+\lambda & 0 & 0 & \gamma \\ 0 & \gamma & -i\gamma & 0 \\ 0 & i\gamma & \gamma & 0 \\ \gamma & 0 & 0 & \left(1-\sqrt{1-\lambda'-\gamma}\right)^2+\lambda \end{pmatrix} \tag{5.18}$$

5.4 One-Qubit Pauli Channel

Kraus Representation: Following Definition 1.7, the Kraus operators will have the form

$$A_k = \sqrt{p_k}P_k \tag{5.19}$$

Therefore, the Kraus operators are

$$A_1 = \sqrt{1-p'}I, \quad A_2 = \sqrt{p_x}X, \quad A_3 = \sqrt{p_y}Y, \quad A_4 = \sqrt{p_z}Z \tag{5.20}$$

Stinespring Representation:

$$V = \begin{pmatrix} \sqrt{1-p'} & 0 \\ 0 & \sqrt{p_x} \\ 0 & -i\sqrt{p_y} \\ \sqrt{p_z} & 0 \\ 0 & \sqrt{1-p'} \\ \sqrt{p_x} & 0 \\ i\sqrt{p_y} & 0 \\ 0 & -\sqrt{p_z} \end{pmatrix} \tag{5.21}$$

Superoperator Representation:

$$[\mathcal{E}] = \begin{pmatrix} (1-p')+p_z & 0 & 0 & p_x+p_y \\ 0 & (1-p')-p_z & p_x-p_y & 0 \\ 0 & p_x-p_y & (1-p')-p_z & 0 \\ p_x+p_y & 0 & 0 & (1-p')+p_z \end{pmatrix} \tag{5.22}$$

Pauli Transfer Matrix:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\left(p_y+p_z\right) & 0 & 0 \\ 0 & 0 & 1-2\left(p_x+p_z\right) & 0 \\ 0 & 0 & 0 & 1-2\left(p_x+p_y\right) \end{pmatrix} \tag{5.23}$$

Choi Representation:

$$\Phi^{\mathcal{E}} = \begin{pmatrix} (1-p')+p_z & 0 & 0 & (1-p')-p_z \\ 0 & p_x+p_y & p_x-p_y & 0 \\ 0 & p_x-p_y & p_x+p_y & 0 \\ (1-p')-p_z & 0 & 0 & (1-p')+p_z \end{pmatrix} \tag{5.24}$$

χ-Matrix Representation:

$$\chi = \begin{pmatrix} 1-p' & 0 & 0 & 0 \\ 0 & p_x & 0 & 0 \\ 0 & 0 & p_y & 0 \\ 0 & 0 & 0 & p_z \end{pmatrix} \tag{5.25}$$

5.5 Depolarizing Channel

As seen in Example 1.3.1 the depolarizing channel is a Pauli Channel with probabilities $p_x = p_y = p_z = p/4$.

Kraus Representation: The Kraus operators are

$$A_1 = \sqrt{1 - \frac{3p}{4}} I, \quad A_2 = \sqrt{\frac{p}{4}} X, \quad A_3 = \sqrt{\frac{p}{4}} Y, \quad A_4 = \sqrt{\frac{p}{4}} Z \tag{5.26}$$

Stinespring Representation:

$$V = \begin{pmatrix} \sqrt{1 - \frac{3}{4}p} & 0 \\ 0 & \sqrt{p/4} \\ 0 & -i\sqrt{p/4} \\ \sqrt{p/4} & 0 \\ 0 & \sqrt{1 - \frac{3}{4}p} \\ \sqrt{p/4} & 0 \\ i\sqrt{p/4} & 0 \\ 0 & -\sqrt{p/4} \end{pmatrix} \tag{5.27}$$

Superoperator Representation:

$$[\mathcal{E}] = \begin{pmatrix} 1 - p/2 & 0 & 0 & p/2 \\ 0 & 1 - p & 0 & 0 \\ 0 & 0 & 1 - p & 0 \\ p/2 & 0 & 0 & 1 - p/2 \end{pmatrix} \tag{5.28}$$

Pauli Transfer Matrix:

$$[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - p & 0 & 0 \\ 0 & 0 & 1 - p & 0 \\ 0 & 0 & 0 & 1 - p \end{pmatrix} \tag{5.29}$$

Choi Representation:

$$\Phi^{\mathcal{E}} = \begin{pmatrix} 1-p/2 & 0 & 0 & 1-p \\ 0 & p/2 & 0 & 0 \\ 0 & 0 & p/2 & 0 \\ 1-p & 0 & 0 & 1-p/2 \end{pmatrix} \tag{5.30}$$

χ-Matrix Representation:

$$\chi = \begin{pmatrix} 1-\frac{3}{4}p & 0 & 0 & 0 \\ 0 & p/4 & 0 & 0 \\ 0 & 0 & p/4 & 0 \\ 0 & 0 & 0 & p/4 \end{pmatrix} \tag{5.31}$$

We can see the relations between certain representations that were previously mentioned in Section 3. The Superoperator and Choi matrices for a superoperator $\mathcal{E}$ are related by the entry permutation as shown in Equation 3.8.

Additionally, we can observe that the entries of the PTM can be represented as a linear combination of the entries in the χ-Matrix. More evidently, the first entry of every PTM is equal to 1 as all the maps are trace-preserving as follows from Theorem 2.6, which is equal to the sum of the diagonal entries of their respective $\chi-$Matrices.

We can also see by Theorem 2.6 that the only CPTP maps that are not unital are the Amplitude Damping Channel (5.1) and the Phase + Amplitude Damping Channel (5.3)

References

[1] Joseph Emerson. *Theory of Quantum Systems: From Mathematical Foundations to Experimental Methods with Applications to Quantum Computing*. 2021.

[2] John Watrous. *The Theory of Quantum Information*. Cambridge University Press, 1 edition, April 2018. ISBN 978-1-316-84814-2 978-1-107-18056-7. doi: 10.1017/9781316848142. URL https://www.cambridge.org/core/product/identifier/9781316848142/type/book.

[3] Joydip Ghosh, Austin G. Fowler, and Michael R. Geller. Surface code with decoherence: An analysis of three superconducting architectures. *Physical Review A*, 86 (6):062318, December 2012. ISSN 1050-2947, 1094-1622. doi: 10.1103/PhysRevA.86.062318. URL http://arxiv.org/abs/1210.5799. arXiv:1210.5799 [cond-mat, physics:quant-ph].

[4] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[5] Tomás Basile and Carlos Pineda. Quantum simulation of Pauli channels and dynamical maps: Algorithm and implementation. *PLOS ONE*, 19(4):e0297210, April 2024. ISSN 1932-6203. doi: 10.1371/journal.pone.0297210. URL https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0297210. Publisher: Public Library of Science.

[6] John Watrous. Quantum channels. URL https://learning.quantum.ibm.com/course/general-formulation-of-quantum-information/quantum-channels. Accessed: 2024-06-13.

[7] Junan Lin, Brandon Buonacorsi, Raymond Laflamme, and Joel J Wallman. On the freedom in representing quantum operations. *New Journal of Physics*, 21(2): 023006, February 2019. ISSN 1367-2630. doi: 10.1088/1367-2630/ab075a. URL https://iopscience.iop.org/article/10.1088/1367-2630/ab075a.

[8] Jonas Helsen, Francesco Battistel, and Barbara M. Terhal. Spectral quantum tomography. *npj Quantum Information*, 5(1):1–11, September 2019. ISSN 2056-6387. doi: 10.1038/s41534-019-0189-0. URL https://www.nature.com/articles/s41534-019-0189-0. Publisher: Nature Publishing Group.

[9] Constantino Rodriguez-Ramos and Colin M Wilmott. On the convex characterisation of the set of unital quantum channels. *Journal of Physics A: Mathematical and Theoretical*, 56(45):455308, November 2023. ISSN 1751-8113, 1751-8121. doi: 10.1088/1751-8121/acfddb. URL https://iopscience.iop.org/article/10.1088/1751-8121/acfddb.

[10] G. Homa, A. Ortega, and M. Koniorczyk. Choi representation of completely positive maps: a technical introduction, February 2024. URL http://arxiv.org/abs/2402.12944. arXiv:2402.12944 [quant-ph].

[11] Gilad Gour. *Introduction to Quantum Information - MATH 667 Course Notes, University of Calgary*. January 2016. URL https://markwgirard.github.io/coursework/QI/QI_main.pdf.

[12] Daniel Greenbaum. Introduction to Quantum Gate Set Tomography, September 2015. URL http://arxiv.org/abs/1509.02921. arXiv:1509.02921 [quant-ph].

[13] Christopher J. Wood, Jacob D. Biamonte, and David G. Cory. Tensor networks and graphical calculus for open quantum systems, May 2015. URL http://arxiv.org/abs/1111.6950. arXiv:1111.6950 [quant-ph].

[14] Forest Benchmarking. Superoperator Representations. URL https://forest-benchmarking.readthedocs.io/en/latest/superoperator_representations.html#igst. Accessed: 2024-07-04.